Asymptotic Distribution of Integers with certain Prime Factorizations

Lenny Neyt

Department of Mathematics
University of Ghent

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Origins: the Matula numbers

The **Matula number** $M(T)$ of a *finite rooted tree* T is inductively defined as

$$M(\bullet) = 1,$$

and for the finite rooted trees $T_1, T_2, \cdots, T_n$ we set

$$M(\bullet(T_1, T_2, \cdots, T_n)) = \prod_{i=1}^{n} p_{M(T_i)},$$

where $p_1 < p_2 < \cdots < p_\nu < \cdots$ is the sequence of prime numbers.

Theorem [Matula, 1968]

The function M is a bijection between the finite rooted trees (up to an isomorphism) and $\mathbb{N}_{\geq 1}$.

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what is the asymptotic behaviour of the counting function $M_h(x)$?

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- $M_1 = \{2^i : i \in \mathbb{N}\}$
- $M_2 = \{p_{2^{i_1}} \cdot \ldots \cdot p_{2^{i_n}} : \text{ each of the } i_j \in \mathbb{N}\}$

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In general

$$M_{n+1} = \{p_{a_1} \cdot \ldots \cdot p_{a_n} : a_i \in M_n\}.$$

Theorem [H. Vernaeve, J. Vindas, A. Weiermann, 2014]

$$M_2(x) \sim \frac{e^{K_2}\sqrt{3}}{2\pi^2}(\log x)^{\frac{\log\left(\frac{\pi}{\sqrt{6\log 2}}\right)}{2\log 2}} \exp\left(\pi\sqrt{\frac{2\log x}{3\log 2}} - \frac{(\log\log x)^2}{8\log 2}\right)$$

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Consider for any $d \geq 1$ the set

$$M_{3,d} := \{M(T) \mid T \text{ is a finite rooted tree of height } 3 \text{ and at most } d \text{ leafs on each end}\}$$

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Goal of this Presentation

Find a strong asymptotic formula for $M_{3,d}(x)$

In General

Let $1 < \lambda_1 < \cdots < \lambda_d$ be d different integers and set

$$A_\Lambda := \{\lambda_1^{a_1} \cdots \lambda_d^{a_d} : a_i \in \mathbb{N} \text{ for } 1 \leq i \leq d\}$$

Consider

$$\Omega_\Lambda := \{p_{a_1} p_{a_2} \cdots p_{a_n} : a_i \in A_\Lambda\}$$

We will find an asymptotic formula for the counting function

$$P_\Lambda(x) = \sum_{\substack{n \in \Omega_\Lambda \\ \log n \leq x}} 1$$

Then in our specific example

$$\lambda_i = p_{2^{i-1}}$$

$$M_{3,d}(x) = P_\Lambda(\log x) + 1$$

Theorem [Ingham, 1941]

Let $A(x)$ be a non-decreasing function, with $A(x) = 0$ for $x \leq 0$. Suppose that the integral

$$f(s) = \int_0^\infty e^{-sx} dA(x),$$

exists for all $\Re(s) > 0$. If there exist certain functions $\varphi(s)$ and $\chi(s)$ s.t.

(i) $f(x) \sim \chi(x) e^{\varphi(x)}$ as $x \to 0+$

(ii) $f(s) = \mathcal{O}\left(\chi(|s|) e^{\varphi(|s|)}\right)$ as $s \to 0+$ uniformly in any Stolz angle

(iii) $-x^k \varphi'(x) \to 0+$ as $x \to 0+$

(iv) some extra conditions on φ and χ

Then

$$A(x) \sim \frac{\chi(\sigma(x)) e^{\varphi(\sigma(x)) + x\sigma(x)}}{\left(2\pi\sigma^2(x)\varphi''(\sigma(x))\right)^{\frac{1}{2}}},$$

as $x \to \infty$, where σ is the solution to

$$-\varphi'(\sigma) = x.$$

An asymptotic for the Laplace transform

$$F_\Lambda(s) := \int_0^\infty e^{-sx} dP_\Lambda(x) = \prod_{a \in A_\Lambda} \frac{1}{1 - e^{-s \log p_a}},$$

So that

$$f_\Lambda(s) := \log F_\Lambda(s) = -\sum_{a \in A_\Lambda} \log\left(1 - e^{-s \log p_a}\right) = \sum_{a \in A_\Lambda} \sum_{k=1}^{\infty} \frac{e^{-sk \log p_a}}{k}$$

$$= \sum_{k < 1/s} \frac{G_\Lambda(ks)}{k} + \mathfrak{c}_r + o(1),$$

with

$$G_\Lambda(s) := \sum_{a \in A_\Lambda} e^{-s \log p_a} = \int_0^\infty e^{-sx} dS_\Lambda(x)$$

and S_Λ the counting function of $\{\log p_a : a \in A_\Lambda\}$.

Lemma

Suppose that for some continuous ψ we have $G_\Lambda(s) - \psi(s) = \mathcal{O}(s)$ as $s \to 0+$, then

$$f_\Lambda(s) = \sum_{k<1/s} \frac{\psi(ks)}{k} + \mathfrak{c} + o(1)$$

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$$f_\Lambda(s) = \sum_{k<1/s} \frac{\psi(ks)}{k} + \mathfrak{c} + o(1)$$

We will show that the asymptotic of $G_\Lambda(s)$ takes the form

$$G_\Lambda(s) = \frac{(d-1)!M_{d-1}}{s^d} + \sum_{k=0}^{d-1}\sum_{l=0}^{d-k} \alpha_{k,l,d}\frac{\log^l s}{s^k} + \sum_{l=1}^{d} \beta_{l,d} s \log^l s + \mathcal{O}(s)$$

by approximating the primes using Cipolla's formula.

Theorem [Cipolla, 1902]

For any $d \geq 1$

$$\log p_\nu = J_{d+1}(\log \nu) + o\left(\frac{1}{\log^{d+1} \nu}\right)$$

where

$$J_{d+1}(x) = x + \log x + \sum_{k=1}^{d+1} \frac{(-1)^{k-1} Q_k(\log x)}{x^k}$$

with the Q_k polynomials of degree k.

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Corollary

$$G_\Lambda(s) = \int_0^\infty \exp\left(-sJ_{d+1}(x)\right) dN_{\log A_\Lambda}(x) + \mathcal{O}(s)$$

Consider, for $0 < t < \infty$, the family of functions

$$J_{d+1}^t(x) = J_{d+1}\left(\frac{x}{t}\right),$$

then

$$\begin{aligned} G_\Lambda(s) &= \int_0^\infty \exp\left(-sJ_{d+1}^s(sx)\right) dN_{\log A_\Lambda}(x) + \mathcal{O}(s) \\ &= \frac{1}{s}\int_0^\infty \exp\left(-sJ_{d+1}^s(x)\right) dN_{\log A_\Lambda}(x/s) + \mathcal{O}(s) \end{aligned}$$

One can show that $\{\exp(-sJ_{d+1}^s(x)) : 0 < t < \infty\}$ is a bounded family in $\mathcal{S}(\mathbb{R})$.

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By the Banach-Steinhaus theorem, a distributional asymptotic formula for $dN_{\log A_\Lambda}(\lambda x)$ as $\lambda \to \infty$, will infer an asymptotic formula for $G_\Lambda(s)$ as $s \to 0+$.

The Laplace transform of $dN_{\log A_\Lambda}$

$$\mathcal{L}\{dN_{\log A_\Lambda}; s\} = \int_0^\infty e^{-sx} dN_{\log A_\Lambda} = \prod_{n=1}^{d} (1 - e^{-s \log \lambda_n})^{-1}$$

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The singularities of each term are exactly $\left\{ \frac{2\pi i k}{\log \lambda_n} : k \in \mathbb{Z} \right\}$, so that the Laplace transform has an isolated singularity at the origin. Close to the origin we have

$$\frac{1}{1 - e^{-s \log \lambda_n}} = \frac{1}{s \log \lambda_n} + \sum_{k=0}^{\infty} \frac{(-1)^{k+1} B_{k+1} \log^k \lambda_n}{(k+1)!} s^k$$

so that

$$\mathcal{L}\{dN_{\log A_\Lambda}; s\} = \sum_{n=0}^{d-1} \frac{M_n n!}{s^{n+1}} + \mathcal{H}(s)$$

with $\mathcal{H}(s)$ analytic in a zero neighbourhood.

Theorem

Take any $f \in \mathcal{S}'[0,\infty)$. Suppose that for some $r > 0$, there exists a continuous function $\mathcal{H}$ on $B(0,r) \cap \{z : \Re(z) > 0\}$ and $d \geq 1$ such that

$$\mathcal{L}\{f;s\} = \sum_{n=0}^{d-1} \frac{M_n n!}{s^{n+1}} + \mathcal{H}(s), \qquad \text{for all } s \in B(0,r) \cap \{z : \Re(z) > 0\}$$

for certain $M_n \in \mathbb{C}$. Suppose that $\mathcal{H}$ extends to some open interval containing the origin on the imaginary line, and is k-times continuously differentiable there, with $k \in \mathbb{N} \cup \{\infty\}$, then

$$f(\lambda x) \sim \sum_{n=0}^{d-1} M_n \lambda^n x_+^n + \sum_{n=0}^{k-1} \frac{(-1)^n \mu_n \delta^{(n)}(x)}{n! \lambda^{n+1}}$$

as $\lambda \to \infty$ in $\mathcal{S}'(\mathbb{R})$, where

$$\mu_n = (i)^n \left(\frac{d^n}{dy^n} \mathcal{H}(iy) \right) \Big|_{y=0}$$

Via the Banach-Steinhaus theorem we get

$$
\begin{aligned}
G_{\Lambda}(s) &= \sum_{n=0}^{d-1} \frac{M_n}{s^{n+1}} \int_0^{\infty} x^n \exp\left(-sJ_{d+1}^{s}(x)\right) dx + \mu_0 + \mathcal{O}(s) \\
&= \sum_{n=0}^{d-1} M_n \int_0^{\infty} x^n \exp\left(-sJ_{d+1}(x)\right) dx + \mu_0 + \mathcal{O}(s) \\
&= s \sum_{n=0}^{d-1} \frac{M_n}{n-1} \int_{\mathrm{Li}^{-1}(1)}^{\infty} \log^{n+1}(\mathrm{Li}(u)) u^{-s-1} du + \mu_0 + \mathcal{O}(s)
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After some calculations

$$G_\Lambda(s) = \frac{(d-1)! M_{d-1}}{s^d} + \sum_{k=0}^{d-1} \sum_{l=0}^{d-k} \alpha_{k,l,d} \frac{\log^l s}{s^k} + \sum_{l=1}^{d} \beta_{l,d} s \log^l s + \mathcal{O}(s)$$

Summing over each term individually provides

$$f_\Lambda(t) = \sum_{k<1/t} \frac{G_\Lambda(kt)}{k} + \mathfrak{c}_r + o(1)$$

$$= \frac{\zeta(d+1)(d-1)!M_{d-1}}{t^d} + \sum_{k=1}^{d-1}\sum_{l=0}^{d-k} \delta_{k,l,d}\frac{\log^l(t)}{t^k}$$

$$+ \sum_{l=1}^{d+1} \epsilon_{l,d}\log^l(t) + \log\chi_c + o(1)$$

$$= \varphi(t) + \log\chi(t) + o(1)$$

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Corollary

$$F_\Lambda(t) \sim \chi(t)e^{\varphi(t)}$$

We may now use Ingham's Tauberian Theorem

Theorem

$$P_{\Lambda}(x) \sim \frac{\chi(\sigma(x))e^{\varphi(\sigma(x))+x\sigma(x)}}{\left(2\pi\sigma^2(x)\varphi''(\sigma(x))\right)^{1/2}}$$

where σ is (for sufficiently large x) the unique solution to

$$-\varphi'(\sigma) = x$$

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Is there a sufficiently precise asymptotic formula for σ consisting only of "familiar" functions?

Yes! By the use of an iterative algorithm.

First observation

Since

$$-\varphi'(t) \sim \frac{\zeta(d+1)d!M_{d-1}}{t^{d+1}}$$

it follows that

$$\sigma(x) \sim \left(\frac{\zeta(d+1)d!M_{d-1}}{x}\right)^{\frac{1}{d+1}}$$

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Corollary

$$P_\Lambda(x) \sim \widetilde{\chi}(x)x^{-\frac{d}{2d+2}}e^{\varphi(\sigma(x))+x\sigma(x)}$$

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Lemma

If $\sigma(x) = \psi(x) + o\left(x^{-1}\right)$ then

$$P_\Lambda(x) \sim \widetilde{\chi}(x)x^{-\frac{d}{2d+2}}e^{\varphi(\psi(x))+x\psi(x)}$$

In general

Consider functions of the type

$$\widetilde{f}(x) = \frac{c}{x^{N+1}} + \sum_{n=1}^{N} x^{-n} \sum_{k=0}^{K} \widetilde{c}_{n,k} \log^k x$$

with $c \neq 0$, and consider (for sufficiently large x) the unique solution of

$$\widetilde{f}(\widetilde{\sigma}_{\widetilde{f}}) = x$$

To ease calculations, we will work with

$$f(x) = \widetilde{f}\left(\frac{1}{x}\right) = cx^{N+1} + \sum_{n=1}^{N} x^n \sum_{k=0}^{K} c_{n,k} \log^k x$$

and

$$\sigma_f(x) = \frac{1}{\widetilde{\sigma}_{\widetilde{f}}(x)}$$

Lemma

If

$$\sigma_f(x) = \psi(x) + o\left(x^{-\frac{N-1}{N+1}}\right)$$

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Plug $\sigma_f(x)$ into f, which infers the identity

$$x = f(\sigma_f) = c\sigma_f^{N+1} + \sum_{n=1}^{N} \sigma_f^n \sum_{k=0}^{K} c_{n,k} \log^k \sigma_f$$

or

$$\sigma_f = \left(\frac{x}{c}\right)^{\frac{1}{N+1}} \left[1 - \sum_{n=1}^{N} \frac{\sigma_f^n}{x} \sum_{k=0}^{K} c_{n,k} \log^k \sigma_f\right]^{\frac{1}{N+1}}$$

Put

$$\eta(x) = c\sigma_f^{N+1}(x) - x = -\sum_{n=1}^{N} \sigma_f^n \sum_{k=0}^{K} c_{n,k} \log^k \sigma_f$$

then the binomial series gives us

$$\begin{aligned}
\sigma_f &= \left(\frac{x}{c}\right)^{\frac{1}{N+1}} \left[1 - \frac{\eta(x)}{x}\right]^{\frac{1}{N+1}} \\
&= \left(\frac{x}{c}\right)^{\frac{1}{N+1}} + \left(\frac{x}{c}\right)^{\frac{1}{N+1}} \sum_{i=1}^{\infty} \binom{(N+1)^{-1}}{i} \frac{\eta^i(x)}{x^i} \\
&= \left(\frac{x}{c}\right)^{\frac{1}{N+1}} + \left(\frac{x}{c}\right)^{\frac{1}{N+1}} \sum_{i=1}^{N} \binom{(N+1)^{-1}}{i} \frac{\eta^i(x)}{x^i} + o\left(x^{-\frac{N-1}{N+1}}\right)
\end{aligned}$$

Define the operator $\kappa_f : \mathcal{C}(0,\infty) \setminus 0 \longrightarrow \mathcal{C}(0,\infty)$ as

$$\kappa_f(\phi)(x) =$$
$$\left(\frac{x}{c}\right)^{\frac{1}{N+1}} \sum_{l=0}^{N} (-1)^l \binom{(N+1)^{-1}}{l} \left(\frac{\sum_{n=1}^{N} \phi(x)^n \sum_{k=0}^{K} c_{n,k} \log^k \phi(x)}{x} \right)^l$$

and let $\kappa_f^{(i)}$ be the *i*th iteration of κ_f.

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and let $\kappa_f^{(i)}$ be the *i*th iteration of κ_f.

Proposition

If for some integer $-1 \leq l \leq N-1$

$$\sigma_f(x) = \psi(x) + o\left(x^{-\frac{l}{N+1}} \log^{(l+1)K}(x)\right)$$

then

$$\sigma_f(x) = \kappa_f(\psi)(x) + o\left(\max\left\{x^{-\frac{N-1}{N+1}}, x^{-\frac{l+1}{N+1}} \log^{(l+2)K}(x)\right\}\right)$$

Conclusion

If we define the functions

$$\tau_f(x) = \kappa_f^N \left(\left(\frac{\cdot}{c} \right)^{\frac{1}{N+1}} \right)(x)$$

$$\tau_{\widetilde{f}}(x) = (\tau_f(x))^{-1}$$

then we have shown

Theorem

As x goes to infinity

$$\sigma_f(x) = \tau_f(x) + o\left(x^{-\frac{N-1}{N+1}}\right)$$

and

$$\widetilde{\sigma}_{\widetilde{f}}(x) = \tau_{\widetilde{f}}(x) + o\left(x^{-1}\right)$$

Final result

For any $d \geq 1$, let $\Lambda = \{p_1, p_2, \ldots, p_{2^d}\}$ and set

$$\sigma_\Lambda(x) := \tau_{-\varphi'}(\log x)$$

Theorem

$$M_{3,d}(x) \sim \widetilde{\chi}(\log x)\,(\log x)^{-\frac{d+1}{2d+4}} \exp\left(\varphi(\sigma_\Lambda(x)) + x\sigma_\Lambda(x)\right)$$