Asymptotic boundedness in ultradistribution spaces

Lenny Neyt

Department of Mathematics: Analysis, Logic and Discrete Mathematics
University of Ghent

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Introduction (1)

General Question

Let $\mathcal{F}$ be a space of translation invariant smooth functions. Can we characterize, for a certain weight function $\omega : \mathbb{R}^d \to \mathbb{R}_+$, when for some $f \in \mathcal{F}'$

$$f * \varphi(h) = O(\omega(h)), \qquad \text{as } |h| \to \infty, \quad \forall \varphi \in \mathcal{F} \tag{1}$$

and

$$f * \varphi(h) = o(\omega(h)), \qquad \text{as } |h| \to \infty, \quad \forall \varphi \in \mathcal{F} \tag{2}$$

- In his book *Théorie des distributions* L. Schwartz considered the case $\mathcal{F} = \mathcal{D}(\mathbb{R}^d)$, $\omega = 1$. He showed that for any $f \in \mathcal{D}'(\mathbb{R}^d)$:
  - f satisfies (1) iff $f \in \mathcal{B}'(\mathbb{R}^d) := (\mathcal{D}_{L^1}(\mathbb{R}^d))'$
  - f satisfies (2) iff $f \in \dot{\mathcal{B}}'(\mathbb{R}^d) := \overline{\mathcal{D}(\mathbb{R}^d)}^{\mathcal{B}'(\mathbb{R}^d)}$
- Dimovski, Pilipović, Prangoski and Vindas considered ultradistributional analogs by use of the parametrix method

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Introduction (2)

In this talk we will consider the case $\mathcal{F}' = \mathcal{S}'^{*}_{\dagger}(\mathbb{R}^d)$ of *tempered ultradistributions*. By exploiting the mapping properties of the short-time Fourier transform (STFT) we are able to establish the characterization under considerably less restrictions on the weight sequences than previously known.

This talk is based on the following two articles:

A. Debrouwere, L. Neyt and J. Vindas, *On the space of ultradistributions vanishing at infinity*, arXiv: 1907.04205.

L. Neyt and J. Vindas, *Asymptotic boundedness and moment asymptotic expansion in ultradistribution spaces*, arXiv: 1906.06232.

Weight sequences

- Let $(M_p)_{p\in\mathbb{N}}$ be a sequence of positive real numbers. We shall always assume that M_p satisfies

$(M.1)$ $M_p^2 \leq M_{p-1}M_{p+1}$, $p \geq 1$

$(M.2)'$ $M_{p+1} \leq AH^pM_p$, $p \in \mathbb{N}$, for some $A, H > 0$

In some cases we also assume the condition

$(M.3)'$ $\sum_{p=0}^{\infty} M_p/M_{p+1} < \infty$

- The associated function of M_p is defined as

$$M(t) := \sup_{p\in\mathbb{N}} \log \frac{t^pM_0}{M_p}, \quad t > 0, \quad M(0) := 0$$

- Example: $M_p = p!^\sigma$, $\sigma > 0$; then $M(t) \asymp t^{1/\sigma}$
  If $\sigma > 1$ this weight sequence satisfies $(M.3)'$
- $M_p \subset N_p$ means that $M_p \leq CL^pN_p$, $p \in \mathbb{N}$, for some $C, L > 0$

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Gelfand-Shilov spaces

Let M_p and A_p be two weight sequences. Set

$$\mathcal{S}^{M_p,\ell}_{A_p,q}(\mathbb{R}^d) := \{\varphi \in C^\infty(\mathbb{R}^d) : \sup_{\alpha\in\mathbb{N}^d}\sup_{x\in\mathbb{R}^d} \frac{|\varphi^{(\alpha)}(x)|e^{A(q|x|)}}{\ell^{|\alpha|}M_{|\alpha|}} < \infty\}$$

We then define

$$\mathcal{S}^{(M_p)}_{(A_p)}(\mathbb{R}^d) = \varprojlim_{\ell\to 0^+} \mathcal{S}^{M_p,\ell}_{A_p,1/\ell}(\mathbb{R}^d), \qquad \mathcal{S}^{\{M_p\}}_{\{A_p\}}(\mathbb{R}^d) = \varinjlim_{\ell\to\infty} \mathcal{S}^{M_p,\ell}_{A_p,1/\ell}(\mathbb{R}^d)$$

The duals $\mathcal{S}'^{(M_p)}_{(A_p)}(\mathbb{R}^d)$ and $\mathcal{S}'^{\{M_p\}}_{\{A_p\}}(\mathbb{R}^d)$ are called *tempered ultradistributions of Beurling and Roumieu type*, respectively. We denote $* = (M_p)$ or $\{M_p\}$ and $\dagger = (A_p)$ or $\{A_p\}$, e.g. $\mathcal{S}'^{*}_{\dagger}(\mathbb{R}^d)$.
We denote by $\mathcal{D}^*(\mathbb{R}^d)$ the subset of compactly supported elements in $\mathcal{S}^*_\dagger(\mathbb{R}^d)$, which is non-trivial and dense if M_p satisfies $(M.3)'$

Assumption

We always assume $\mathcal{S}^{(M_p)}_{(A_p)}(\mathbb{R}^d)$ is non-trivial. A sufficient condition is, e.g, that $p!^\sigma \subset M_p$ and $p!^\tau \subset A_p$ for $\sigma + \tau > 1$

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short-time Fourier transform (1)

Take some $\psi \in \mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d) \setminus \{0\}$. For any $f \in \mathcal{S}_\dagger'^*(\mathbb{R}^d)$ we define the STFT $V_\psi f$ as

$$V_\psi f(x,\xi) = \left\langle f, e^{-2\pi i \xi \cdot} \overline{\psi(\cdot - x)} \right\rangle = \int_{\mathbb{R}^d} f(t) \overline{\psi(t-x)} e^{-2\pi i \xi \cdot t} dt$$

For a weight sequence N_p we denote by $\mathcal{S}_{(N_p)}(\mathbb{R}^d)$ the Fréchet space of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that

$$\sup_{|\alpha| \le k} \sup_{x \in \mathbb{R}^d} |\varphi^{(\alpha)}(x)| e^{N(k|x|)} < \infty, \qquad \forall k \in \mathbb{N}.$$

Proposition

The following mappings are continuous

$$V_\psi : \mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d) \to \mathcal{S}_{(A_p)}(\mathbb{R}^d_x) \widehat{\otimes} \mathcal{S}_{(M_p)}(\mathbb{R}^d_\xi)$$

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short-time Fourier transform (2)

We define the adjoint STFT of $F \in \mathcal{S}'_{(A_p)}(\mathbb{R}^d_x)\widehat{\otimes}\mathcal{S}'_{(M_p)}(\mathbb{R}^d_\xi)$ as

$$\left\langle V_\psi^* F, \varphi \right\rangle := \left\langle F, \overline{V_{\overline{\psi}}\varphi} \right\rangle, \qquad \varphi \in \mathcal{S}^{(M_p)}_{(A_p)}(\mathbb{R}^d)$$

Proposition

Let $\gamma \in \mathcal{S}^{(M_p)}_{(A_p)}(\mathbb{R}^d)$ be a synthesis window for ψ, i.e. $(\gamma, \psi)_{L^2} \neq 0$, then the reconstruction formula

$$\frac{1}{(\gamma, \psi)_{L^2}} V_\gamma^* \circ V_\psi = \mathrm{id}_{\mathcal{S}'^{(M_p)}_{(A_p)}(\mathbb{R}^d)}$$

is valid and the desingularization formula

$$\langle f, \varphi \rangle = \frac{1}{(\gamma, \psi)_{L^2}} \int\int_{\mathbb{R}^{2d}} V_\psi f(x,\xi) V_{\overline{\gamma}}\varphi(x, -\xi) dx d\xi$$

holds for all $f \in \mathcal{S}'^*_\dagger(\mathbb{R}^d)$ and $\varphi \in \mathcal{S}^*_\dagger(\mathbb{R}^d)$

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$$\frac{1}{(\gamma, \psi)_{L^2}} V_\gamma^* \circ V_\psi = \mathrm{id}_{\mathcal{S}'^{(M_p)}_{(A_p)}(\mathbb{R}^d)}$$

is valid and the desingularization formula

$$\langle f, \varphi \rangle = \frac{1}{(\gamma, \psi)_{L^2}} \int\int_{\mathbb{R}^{2d}} V_\psi f(x,\xi) V_{\overline{\gamma}}\varphi(x, -\xi) dx d\xi$$

holds for all $f \in \mathcal{S}'^*_\dagger(\mathbb{R}^d)$ and $\varphi \in \mathcal{S}^*_\dagger(\mathbb{R}^d)$

The space $\mathcal{D}^*_{L^1_\omega}$

Take $\omega : \mathbb{R}^d \to \mathbb{R}_+$ such that ω and ω^{-1} are locally bounded and is (A_p)-admissible (resp. $\{A_p\}$-admissible), i.e.

$$\exists \lambda > 0 \text{ (resp. } \forall \lambda > 0) \; \exists C > 0 \; \forall x, t \in \mathbb{R}^d : \; \omega(x+t) \leq C\omega(x)e^{A(\lambda|t|)}$$

For $\ell > 0$ we denote by $\mathcal{D}^{M_p,\ell}_{L^1_\omega}$ the Banach spaces of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that

$$\sup_{\alpha \in \mathbb{N}^d} \frac{\int_{\mathbb{R}^d} |\varphi^{(\alpha)}(x)|\omega(x)dx}{\ell^{|\alpha|} M_{|\alpha|}} < \infty$$

We define

$$\mathcal{D}^{(M_p)}_{L^1_\omega} = \varprojlim_{\ell \to 0^+} \mathcal{D}^{M_p,\ell}_{L^1_\omega}, \qquad \mathcal{D}^{\{M_p\}}_{L^1_\omega} = \varinjlim_{\ell \to \infty} \mathcal{D}^{M_p,\ell}_{L^1_\omega}$$

Lemma

We have the following dense continuous inclusion

$$\mathcal{S}^*_\dagger(\mathbb{R}^d) \hookrightarrow \mathcal{D}^*_{L^1_\omega} \hookrightarrow \mathcal{S}'^*_\dagger(\mathbb{R}^d)$$

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The space $\mathcal{B}'^{*}_{\omega}$ (1)

We denote the dual of $\mathcal{D}^{*}_{L^1_\omega}$ by $\mathcal{B}'^{*}_{\omega}$

Theorem 1 (N. and Vindas, 2019)

Let $\psi \in \mathcal{S}^{(M_p)}_{(A_p)}(\mathbb{R}^d) \setminus \{0\}$. For any $f \in \mathcal{S}'^{*}_{\dagger}(\mathbb{R}^d)$ TFAE:

(1) $f \in \mathcal{B}'^{*}_{\omega}$

(2) $f(x+h) = O(\omega(h))$, $h \in \mathbb{R}^d$, in $\mathcal{S}'^{*}_{\dagger}(\mathbb{R}^d)$

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$f \in \mathcal{B}'^{*}_{\omega}$ if and only if there are continuous $\{f_\alpha\}_{\alpha\in\mathbb{N}^d}$ s.t. for some $\ell > 0$ (for any $\ell > 0$) there exits $C_\ell > 0$ s.t.

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and

$$f = \sum_{\alpha\in\mathbb{N}^d} f_\alpha^{(\alpha)}$$

This follows from the Hahn-Banach theorem, where in the Roumieu case we employed the projective description

$$\mathcal{D}^{\{M_p\}}_{L^1_\omega} \cong \varprojlim_{(r_p)\in\mathfrak{R}} D^{M_p \prod_{j=1}^{p} r_j, 1}_{L^1_\omega}$$

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We define $\dot{\mathcal{B}}'^{*}_{\omega}$ as the closure of $C_c(\mathbb{R}^d)$ in $\mathcal{B}'^{*}_{\omega}$. Equivalently, it is the closure of $\mathcal{S}^{*}_{\dagger}(\mathbb{R}^d)$ in $\mathcal{B}'^{*}_{\omega}$.

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The space $\mathcal{D}^{*}_{L^1_\omega}$ is not semi-reflexive, i.e. $(\mathcal{B}'^{*}_{\omega})' \neq \mathcal{D}^{*}_{L^1_\omega}$. Can we find a space whose dual is $\mathcal{D}^{*}_{L^1_\omega}$?

Proposition

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Theorem 4 (Debrouwere, N. and Vindas, 2019)

The canonical inclusion mapping

$$\iota : \mathcal{D}^{*}_{L^1_\omega} \to (\dot{\mathcal{B}}'^{*}_{\omega})'_b : \quad \varphi \mapsto (f \mapsto \langle f, \varphi \rangle)$$

is a topological isomorphism

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Proof of Theorem 5 (1)

Let E be a Banach space. $\Lambda_{M_p,\ell}(E)$ as the space of all $(e_\alpha)_{\alpha\in\mathbb{N}^d} \in E^{\mathbb{N}^d}$ s.t. for some $\ell > 0$ (any $\ell > 0$)

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Claim

The following map is surjective

$$S : \Lambda_*(C_{0,\omega}(\mathbb{R}^d)) \to \dot{\mathcal{B}}'^*_\omega : \quad (f_\alpha)_{\alpha\in\mathbb{N}^d} \mapsto \sum_{\alpha\in\mathbb{N}^d} f_\alpha^{(\alpha)}$$

Lemma

Let $S : E \to F$ be a continuous linear map. Suppose E and $\operatorname{Im} S$ are Mackey and $E/\ker S$ is complete. Then S is surjective if

1. $S^t : F' \to E'$ is injective
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2. $\operatorname{Im} S^t$ is weakly closed in E'

Proof of Theorem 5 (2)

- $E = \Lambda_*(C_{0,\omega}(\mathbb{R}^d))$ is a complete (LB)-space (a Fréchet space) so that E is Mackey and $E/\ker S$ is complete. $\operatorname{Im} S$ is infrabarreled (is Fréchet), whence Mackey
- S^t may be identified with the injective continuous linear mapping

$$\mathcal{D}^*_{L^1_\omega} \to \Lambda'_*(\mathcal{M}^1_\omega): \quad \varphi \mapsto ((-1)^{|\alpha|}\varphi^{(\alpha)})_{\alpha\in\mathbb{N}^d}$$

  $\Lambda'_{(M_p)}(E) := \Lambda_{\{1/M_p\}}(E)$, $\Lambda'_{\{M_p\}}(E) := \Lambda_{(1/M_p)}(E)$ and $\mathcal{M}^1_\omega = (C_{0,\omega}(\mathbb{R}^d))'$
- One shows that $\operatorname{Im} S^t$ is weakly closed in $\Lambda'_*(\mathcal{M}^1_\omega)$ by noting that for any net $(\varphi_j)_j$ in $\mathcal{D}_{L^1_\omega}$ if $\varphi_j^{(\alpha)} \to \mu_\alpha$ weakly in $\mathcal{M}^1_\omega$, $\forall \alpha \in \mathbb{N}^d$, then $\mu_0^{(\alpha)} = \mu_\alpha$ and $\mu_0 \in \mathcal{D}_{L^1_\omega}$

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Applications

The following can now be found under the conditions $(M.1)$, $(M.2)'$ and $(M.3)'$:

1. Structural theorems for S-asymptotic behavior (boundedness) on convex cones in $\mathcal{D}'^*(\mathbb{R}^d)$
2. Structural theorems for quasiasymptotic behavior (boundedness)

$$f(\lambda x) \sim \lambda^q L(\lambda) g(x)$$

   in $\mathcal{D}'^*(\mathbb{R})$, both at infinity and the origin
3. A complete characterization of the moment asymptotic expansion

$$f(\lambda x) \sim \sum_{\alpha \in \mathbb{N}^d} \frac{(-1)^{|\alpha|} \mu_\alpha \delta^{(\alpha)}(x)}{\alpha! \lambda^{|\alpha|+1}}$$

   in $\mathcal{D}'^*(\mathbb{R})$ via the space $\mathcal{K}'^*(\mathbb{R})$
4. A partial characterization of the uniform moment asymptotic expansion in $\mathcal{D}'^*(\mathbb{R})$ via the space $\mathcal{K}'^*_\dagger(\mathbb{R})$

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