On the extension of surjective operators and its applications

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Outline

1. Motivation: the parameter dependence of solutions to PDEs
2. The abstract Mittag-Leffler procedure
3. Extensions of surjective operators between (PLB)-spaces
4. The extension of Eidelheit sequences
5. Application: the vector-valued Stieltjes moment problem

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The parameter dependence of solutions to PDEs

Let $P(D) = \sum_{|\alpha| \leq N} c_\alpha D^\alpha$ be a constant coefficient PDO on $\mathbb{R}^d$. For $X \subseteq \mathbb{R}^d$ open we then find the continuous linear operator

$$P(D) : C^\infty(X) \to C^\infty(X), \quad \varphi \mapsto P(D)\varphi = \sum_{|\alpha| \leq N} c_\alpha D^\alpha \varphi$$

Question

Suppose that $P(D) : C^\infty(X) \to C^\infty(X)$ is surjective. Given data $(\psi_\lambda)_\lambda \subset C^\infty(X)$ depending *regularly* on the parameter λ, do there exist $\varphi_\lambda \in C^\infty(X)$ such that

$$P(D)\varphi_\lambda = \psi_\lambda$$

such that $(\varphi_\lambda)_\lambda$ depends *regularly* on λ?

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The ε-product and regularity

Let E, F be lcHs (= locally convex Hausdorff spaces). We define

$$F\varepsilon E = L_b(F'_c, E)$$

For a continuous linear map $A : G \to H$ between two lcHs G, H, we define the continuous linear map

$$A\varepsilon \operatorname{id}_E : G\varepsilon E \to H\varepsilon E, \quad L \mapsto L \circ A^t$$

Suppose $E = E(\Lambda)$ is a space of numerical functions on a set Λ. We define

$$E(\Lambda; F) = \{(f_\lambda)_{\lambda\in\Lambda} \in F^\Lambda \mid \lambda \mapsto \langle f', f_\lambda\rangle \in E \text{ for all } f' \in F\}$$

Then, under mild assumptions on E we have

$$F\varepsilon E \cong E(\Lambda; F)$$

and under this identification $A\varepsilon \operatorname{id}_E$ is given by

$$E(\Lambda; G) \to E(\Lambda; H), \quad (g_\lambda)_{\lambda\in\Lambda} \mapsto (A(g_\lambda))_{\lambda\in\Lambda}$$

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The parameter dependence of solutions to PDEs – Reformulation

We may now restate the problem of parameter dependence for PDEs exactly as follows

Question

Suppose that $P(D) : C^\infty(X) \to C^\infty(X)$ is surjective. For which lcHs E is the map

$$P(D)\varepsilon \operatorname{id}_E : C^\infty(X)\varepsilon E \to C^\infty(X)\varepsilon E$$

surjective?

The extension of surjective operators

This is a particular instance of the following general question.

Question

Suppose that $A : G \to H$ is a surjective continuous linear map between two lcHs. For which lcHs E is the map

$$A \varepsilon \operatorname{id}_E : G \varepsilon E \to H \varepsilon E$$

surjective?

History

- Grothendieck showed in 1955 that this always holds for nuclear Fréchet spaces G, H and a Fréchet space E.
- Vogt and Wagner showed during the 1970s that this holds for nuclear Fréchet spaces G, H and a Montel (DF)-space E if $\ker A$ satisfies the so-called condition (Ω) and E the so-called condition (A).
  $\ker P(D)$ satisfies (Ω) if
  - X is convex and $P(D)$ is hypoelliptic (Petzsche);
  - $P(D)$ is elliptic (Vogt);
  - $P(D)$ is semi-elliptic with a single characteristic direction, e.g., the heat and Schrödinger equation;
  - $d = 2$ and $P(D)$ is hypoelliptic.
- During the 2000s Bonet and Domanski provided several analogous results in the case of (PLS)-spaces.

Problem

Most of these results require unnecessary topological conditions on the space E.

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Introduction

At the end of the 19th century and beginning of the 20th century a certain technique was often used to show the surjectivity of an operator. In particular, this procedure was used by Mittag-Leffler to show the well-know "Mittag-Leffler theorem". In his memory, we call it the "Mittag-Leffler procedure".

The idea of the Mittag-Leffler procedure consists out of two steps:

1. Solve the problem "locally".
2. "Paste" the local solutions together.

By the second half of the 20th century steps were made to unify all of the special cases into one abstract theorem.

Letter from Grothendieck to Serre in 1954 : *Otherwise, I put in a systematic form, once and for all, the Mittag-Leffler approximation process, which you mentioned to me for a special case. I recommend to Bourbaki's attention the following theorem, which people reprove again and again in all kinds of special cases*.

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Homological algebra – The derived functor (1)

Let $\mathscr{K}$ be a *semi-abelian* category *with many injective objects*.

Then, for every object X in $\mathscr{K}$ there exist *injective* objects $(I_n)_{n\in\mathbb{N}}$ in $\mathscr{K}$ and morphisms $i : X \to I_0$ and $i_n : I_n \to I_{n+1}$ such that

$$0 \longrightarrow X \xrightarrow{i} I_0 \xrightarrow{i_0} I_1 \xrightarrow{i_1} I_2 \xrightarrow{i_2} \dots$$

is exact. The complex is called the injective resolution of X.

Let F be a *covariant semi-injective* functor between $\mathscr{K}$ and an *abelian* category $\mathscr{A}$. We find the complex

$$0 \longrightarrow F(I_0) \xrightarrow{i_0^*} F(I_1) \xrightarrow{i_1^*} F(I_2) \xrightarrow{i_2^*} \dots$$

The derived functors of F are defined as the cohomology of this complex, i.e.,

$$F^0(X) = \ker i_0^*, \qquad F^{n+1}(X) = \operatorname{coker}(\operatorname{im} i_n^* \to \ker i_{n+1}^*).$$

It is independent of the chosen injective resolution for X.

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Homological algebra – The derived functor (2)

Theorem

If

$$0 \longrightarrow Y \xrightarrow{f} X \xrightarrow{g} Z \longrightarrow 0$$

is a short exact sequence in $\mathscr{K}$, then there exists an exact complex in $\mathscr{A}$

$$\begin{array}{ccccccc}
0 & \longrightarrow & F^0(Y) & \longrightarrow & F^0(X) & \longrightarrow & F^0(Z) \\
 & & & \overset{\delta^0}{\swarrow} & & & \\
 & & F^1(Y) & \longrightarrow & F^1(X) & \longrightarrow & F^1(Z) \\
 & & & \overset{\delta^1}{\swarrow} & & & \\
 & & F^2(Y) & \longrightarrow & F^2(X) & \longrightarrow & F^2(Z) \xrightarrow{\delta^2} \dots
\end{array}$$

Homological algebra – The derived functor (3)

Corollary

If $F^1(Y) = 0$, then,

$$0 \longrightarrow F^0(Y) \longrightarrow F^0(X) \longrightarrow F^0(Z) \longrightarrow 0$$

is a short exact sequence. Moreover, if $F^1(X) = 0$, then the converse is also true.

The category of projective spectra

A (countable) projective spectrum $\mathcal{X} = (X_n, \rho_n^m)$ is a sequence of vector spaces $(X_n)_{n\in\mathbb{N}}$ and linear maps $\rho_n^m : X_m \to X_n$, $m \geq n$, such that

$$\rho_n^n = \mathrm{id}_{X_n}, \quad \rho_n^k = \rho_n^m \circ \rho_m^k, \qquad \forall n \leq m \leq k$$

A morphism $f = (f_n)_{n\in\mathbb{N}} : \mathcal{X} \to \mathcal{Z}$ between two projective spectra $\mathcal{X} = (X_n, \rho_n^m)$ and $\mathcal{Z} = (Z_n, \tau_n^m)$ consists of linear maps $f_n : X_n \to Z_n$ such that

$$f_n \circ \rho_n^m = \tau_n^m \circ f_m, \qquad \forall n \leq m$$

Lemma

The class of projective spectra and morphisms forms an *abelian* category which has *sufficiently many injective objects*.

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$$f_n \circ \rho_n^m = \tau_n^m \circ f_m, \qquad \forall n \leq m$$

Lemma

The class of projective spectra and morphisms forms an *abelian* category which has *sufficiently many injective objects*.

The projective functor

For a projective spectrum $\mathcal{X} = (X_n, \rho_m^n)$ we set

$$\operatorname{Proj}\mathcal{X} = \left\{ (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n \mid \rho_n^m(x_m) = x_n \text{ for all } n \leq m \right\}$$

If each X_n is a lcHs we may also endow it with the induced topology of $\prod_{n\in\mathbb{N}} X_n$.

For a morphisms $f = (f_n)_{n\in\mathbb{N}} : \mathcal{X} \to \mathcal{Z}$ we set

$$\operatorname{Proj} f : \operatorname{Proj}\mathcal{X} \to \operatorname{Proj}\mathcal{Z}, \quad (x_n)_{n\in\mathbb{N}} \mapsto (f_n(x_n))_{n\in\mathbb{N}}$$

We also consider the continuous linear mappings

$$\rho_n^\infty : \operatorname{Proj}\mathcal{X} \to X_n, \quad (x_n)_{n\in N} \mapsto x_n, \qquad n \in \mathbb{N}$$

and note that

$$\rho_n^\infty = \rho_n^m \circ \rho_m^\infty, \qquad \forall n \leq m$$

Proj is an *injective* functor between the category of projective spectra and the category of vector spaces, hence its derived functors exist.

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The derived projective functor

Lemma

Consider the mapping

$$\Psi : \prod_{n\in\mathbb{N}} X_n \to \prod_{n\in\mathbb{N}} X_n, \quad (x_n)_{n\in\mathbb{N}} \mapsto (x_n - \rho_n^{n+1}(x_{n+1}))_{n\in\mathbb{N}},$$

then

$$\operatorname{Proj}^0 \mathcal{X} \cong \operatorname{Proj} \mathcal{X} = \ker \Psi, \qquad \operatorname{Proj}^n \mathcal{X} = 0 \text{ for all } n \geq 2,$$

$$\operatorname{Proj}^1 \mathcal{X} \cong \left(\prod_{n\in\mathbb{N}} X_n\right) / \operatorname{im} \Psi$$

Corollary

$\operatorname{Proj}^1 \mathcal{X} = 0$ if and only if

$$\forall (a_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n \; \exists (b_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n : \quad a_n = b_n - \rho_n^{n+1}(b_{n+1})$$

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The abstract Mittag-Leffler theorem (1)

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Suppose

$$0 \longrightarrow \mathcal{Y} \xrightarrow{f} \mathcal{X} \xrightarrow{g} \mathcal{Z} \longrightarrow 0$$

is a short exact sequence of projective spectra. If $\mathrm{Proj}^1 \mathcal{Y} = 0$, then,

$$0 \longrightarrow \mathrm{Proj}\, \mathcal{Y} \xrightarrow{f} \mathrm{Proj}\, \mathcal{X} \xrightarrow{g} \mathrm{Proj}\, \mathcal{Z} \longrightarrow 0$$

is a short exact sequence of vector spaces. Moreover, if $\mathrm{Proj}^1 \mathcal{X} = 0$, then the converse is also true.

The abstract Mittag-Leffler theorem (2)

Suppose

$$Q : X \to Z$$

is a linear map and

$$X \cong \operatorname{Proj} \mathcal{X} = \operatorname{Proj}(X_n, \rho_n^m), \qquad Z \cong \operatorname{Proj} \mathcal{Z} = \operatorname{Proj}(Z_n, \tau_n^m)$$

such that for certain linear maps $Q_n : X_n \to Z_n$

$$\tau_n^\infty \circ Q = Q_n \circ \rho_n^\infty$$

Put $Y_n := \ker Q_n$ and $\varrho_n^m = (\rho_n^m)_{|Y_n}$. Then, $\mathcal{Y} = (Y_n, \varrho_n^m)$ is a projective spectrum such that

$$\ker Q \cong \operatorname{Proj} \mathcal{Y}$$

Corollary

Q is surjective if

$$\forall n \in \mathbb{N}\ \exists m \geq n : \quad \tau_n^m(Z_m) \subseteq Q_n(X_n)$$

and

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A sufficient condition for $\operatorname{Proj}^1 \mathcal{Y} = 0$

There are a variety of conditions that are sufficient, necessary, or even equivalent to $\operatorname{Proj}^1 \mathcal{Y} = 0$, depending on the topological structure of the spaces Y_n. For a thorough overview, we refer to

J. Wengenroth, *Derived Functors in Functional Analysis*, Springer-Verlag, Berlin, 2003.

Here, we will need the following sufficient condition. For a lcHs F we will use the notation

$$\mathscr{BD}(F) = \{B \subseteq F \mid B \text{ is a Banach disk of } F\}$$

Theorem (Langenbruch, 2004)

Suppose $\mathcal{Y} = (Y_n, \varrho_n^m)$ is a projective spectrum where each Y_n is a lcHs and each $\varrho_n^m \in L(Y_m, Y_n)$. If

$$\forall n \in \mathbb{N} \;\; \exists m \geq n \, \forall k \geq m \, \exists B \in \mathscr{BD}(Y_n) \, \forall M \in \mathscr{BD}(Y_m) \, \exists K \in \mathscr{BD}(Y_k)$$
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Outline

1. Motivation: the parameter dependence of solutions to PDEs
2. The abstract Mittag-Leffler procedure
3. Extensions of surjective operators between (PLB)-spaces
4. The extension of Eidelheit sequences
5. Application: the vector-valued Stieltjes moment problem

(PLB)-spaces

A (DF)-space F is a lcHs with a countable fundamental system of bounded sets $\{B_N\}_{N\in\mathbb{N}}$ and such that it is a *countably quasi-barrelled space*. In particular, its strong dual of F'_b is a Fréchet space.

If F is sequentially complete, then F has a countable increasing fundamental system of Banach disks $\{B_N\}_{N\in\mathbb{N}}$, and we find the Banach space

$$F_N = \bigcup_{r>0} rB_N \text{ with norm } \|x\|_N = \inf\{r > 0 : x \in rB_N\}$$

If the embeddings $F_N \to F_{N+1}$ are compact, resp. nuclear, then we say that F is a (DFS)-space, resp. (DFN)-space.

A lcHs F is called a (PLB)-space if $F \cong \operatorname{Proj}\mathcal{F}$ as lcHs for some projective spectrum $\mathcal{F} = \{F_n, \rho_n^m\}$ of sequentially complete (DF)-spaces.
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(PLB)-spaces – Examples

Many spaces appearing in analysis are (PLB)-spaces. For example:

- Every Fréchet space and every sequentially complete (DF)-space is a (PLB)-space;
- The space of distributions $\mathcal{D}'(\Omega)$;
- The space of real analytic functions $\mathscr{A}(\Omega)$;
- The Roumieu class of ultradifferentiable functions $\mathcal{E}^{\{M\}}(\Omega)$;
- The multiplier space $\mathcal{O}_M(\mathbb{R}^d)$ of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$.
- Given a matrix $\mathcal{A} = (a_{n,N})_{n,N\in\mathbb{N}}$ of sequences on some index set I of positive real numbers such that

$$a_{n,N+1,j} \leq a_{n,N,j} \leq a_{n+1,N,j}, \qquad \forall n, N \in \mathbb{N}, j \in I$$

the sequence space

$$k(\mathcal{A}) = \{(c_j)_{j\in I} \in \mathbb{C}^I \mid \forall n\ \exists N : \sup_{j\in I} a_{n,N,j}|c_j| < \infty\}$$

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The ε-product between (PLB)-spaces

Let $G \cong \operatorname{Proj} \mathcal{G} = \operatorname{Proj}(G_n, \rho_n^m)$ and $E \cong \operatorname{Proj} \mathcal{E} = \operatorname{Proj}(E_n, \pi_n^m)$ be two (PLB)-spaces.

We consider the projective spectrum

$$\mathcal{G} \varepsilon \mathcal{E} = (L(G_n', E_n), \kappa_n^m)$$

with

$$\kappa_n^m : L(G_m', E_m) \to L(G_n', E_n), \quad L \mapsto \pi_n^m \circ L \circ (\rho_n^m)^t$$

Then,

$$G \varepsilon E \cong \operatorname{Proj} \mathcal{G} \varepsilon \mathcal{E}$$

as vector spaces.

The ε-product between (PLB)-spaces

Let $G \cong \operatorname{Proj} \mathcal{G} = \operatorname{Proj}(G_n, \rho_n^m)$ and $E \cong \operatorname{Proj} \mathcal{E} = \operatorname{Proj}(E_n, \pi_n^m)$ be two (PLB)-spaces.

We consider the projective spectrum

$$\mathcal{G}\varepsilon\mathcal{E} = (L(G_n', E_n), \kappa_n^m)$$

with

$$\kappa_n^m : L(G_m', E_m) \to L(G_n', E_n), \quad L \mapsto \pi_n^m \circ L \circ (\rho_n^m)^t$$

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The extensions of operators

Proposition

Let

$$0 \longrightarrow F \xrightarrow{\iota} G \xrightarrow{A} H \longrightarrow 0$$

be a short exact sequence of (PLN)-spaces. Let E be a (PLB)-space.

Then, if $\operatorname{Proj}^1 \mathcal{F}\varepsilon\mathcal{E} = 0$, we have the short exact sequence

$$0 \longrightarrow F\varepsilon E \xrightarrow{\iota\varepsilon\,\mathrm{id}} G\varepsilon E \xrightarrow{A\varepsilon\,\mathrm{id}} H\varepsilon E \longrightarrow 0$$

Moreover, if $\operatorname{Proj}^1 \mathcal{G}\varepsilon\mathcal{E} = 0$, this is also necessary.

Sufficient conditions for the vanishing of Proj^1

Lemma

Let $W \cong \operatorname{Proj} \mathcal{W} = \operatorname{Proj}(W_n, \tau_n^m)$ be some (PLN)-space and $q : W \to F$ a surjective continuous linear operator. If $\mathrm{Proj}^1 \mathcal{W}\varepsilon\mathcal{E} = 0$, then also $\mathrm{Proj}^1 \mathcal{F}\varepsilon\mathcal{E} = 0$.

Lemma

If

$$\forall n \in \mathbb{N}\ \exists m \geq n\ \ \forall k \geq m\ \exists N \in \mathbb{N}\ \forall M \geq N, \varepsilon > 0\ \exists K \geq N, R > 0$$

$$\forall \psi \in L(W'_{m,M}, E_{m,M}), \quad \sup_{x \in B_{W',m,M}} \|\psi(x)\|_{E,m,M} \leq 1$$

$$\exists \phi \in L(W'_{k,K}, E_{k,K}), \quad \sup_{x \in B_{W',k,K}} \|\phi(x)\|_{E,k,K} \leq R$$

$$\sup_{x \in B_{W',n,N}} \|\pi_n^m \circ \psi \circ (\tau_n^m)^t(x) - \pi_n^k \circ \phi \circ (\tau_n^k)^t(x)\|_{E,n,N} \leq \varepsilon,$$

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then $\mathrm{Proj}^1 \mathcal{W}\varepsilon\mathcal{E} = 0$.

The dual interpolation estimate

Let $F \cong \operatorname{Proj} \mathcal{F} = \operatorname{Proj}(F_n, \rho_n^m)$ be a (PLB)-space and let $\{B_{n,N}\}_{N\in\mathbb{N}}$ be a fundamental system of bounded sets for F_n.

We say F satisfies the dual interpolation estimate for small θ – $(\mathrm{DIE})_{\theta<}$ if

$$\forall n\, \exists m \geq n\, \forall k \geq m\, \exists N\, \forall M \geq N\, \exists \theta_0 \in (0,1)\, \forall \theta \in (0,\theta_0)\, \exists K \geq M, C > 0$$

$$\forall r > 0: \quad \rho_n^m(B_{m,M}) \subseteq C\left(r^\theta \rho_n^k(B_{k,K}) + \frac{1}{r^{1-\theta}} B_{n,N}\right)$$

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We say F satisfies the dual interpolation estimate for big θ – $(\mathrm{DIE})_{\theta>}$ if

$$\forall n\, \exists m \geq n\, \forall k \geq m\, \exists N\, \forall M \geq N\, \exists \theta_0 \in (0,1)\, \forall \theta \in (\theta_0, 1)\, \exists K \geq M, C > 0$$

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The extension theorem

Theorem

Let G, H be (PLN)-spaces and

$$A : G \to H$$

a surjective continuous linear map.

We suppose that there exists a (PLN)-space $k(\mathcal{A})$ and a surjective continuous linear map $k(\mathcal{A}) \to \ker A$.

Let E be some (PLB)-space and suppose that both $k(\mathcal{A})$ and E both satisfy either $(\mathrm{DIE})_{\theta<}$ or $(\mathrm{DIE})_{\theta>}$.

Then,

$$A\varepsilon \,\mathrm{id} : G\varepsilon E \to H\varepsilon E$$

is surjective.

The conditions (Ω) and (DN)

Let F be a Fréchet space with increasing fundamental system of seminorms $(|\cdot\|_n)_{n\in\mathbb{N}}$.

A Fréchet space is said to satisfy (Ω) if

$$\forall n \in \mathbb{N} \;\; \exists m \geq n \, \forall k \geq m \, \exists \theta \in (0,1), C > 0$$
$$\|y\|_m^* \leq C \|y\|_k^{*\theta} \|y\|_n^{*1-\theta}, \quad \forall y \in F'$$

A nuclear Fréchet space satisfies (Ω) if and only if it is isomorphic to a quotient of $s = k(((1+m)^n)_{m\in\mathbb{N}})$.

A Fréchet space is said to satisfy (DN) if

$$\forall n \in \mathbb{N} \;\; \exists m \geq n \, \forall k \geq m \, \exists C > 0$$
$$\|x\|_m^2 \leq C \|x\|_k \|x\|_n, \quad \forall x \in F$$

A nuclear Fréchet space satisfies (DN) if and only if it is isomorphic to a closed subspace of s.

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The condition (PA)

Let $F \cong \operatorname{Proj} \mathcal{F} = \operatorname{Proj}(F_n, \rho_n^m)$ be a (PLB)-space and let $\{B_{n,N}\}_{N\in\mathbb{N}}$ be a fundamental system of bounded sets for F_n.

We say F satisfies (PA) if

$$\forall n\, \exists m \geq n\, \forall k \geq m\, \exists N\, \forall M \geq N\, \exists \theta_0 \in (0,1)\, \forall \theta \in (\theta_0, 1)\, \exists K \geq M, C > 0$$

$$\forall 0 < r \leq 1: \quad \rho_n^m(B_{m,M}) \subseteq C\left(r^\theta \rho_n^k(B_{k,K}) + \frac{1}{r^{1-\theta}} B_{n,N}\right),$$

A (DF)-space F is said to satisfy (A) if F_b' satisfies (DN).

Then, F satisfies (PA) if and only if it satisfies (A).

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The extension theorem – Fréchet case

Theorem

Let G, H be nuclear Fréchet spaces and

$$A : G \to H$$

a surjective continuous linear map.

We suppose that $\ker A$ satisfies (Ω)

Let E be some (PLB)-space that satisfies (PA).

Then,

$$A \varepsilon \mathrm{id} : G \varepsilon E \to H \varepsilon E$$

is surjective.

Outline

1. Motivation: the parameter dependence of solutions to PDEs
2. The abstract Mittag-Leffler procedure
3. Extensions of surjective operators between (PLB)-spaces
4. The extension of Eidelheit sequences
5. Application: the vector-valued Stieltjes moment problem

Eidelheit sequences

Let G be a lcHs and $(A_\alpha)_{\alpha\in\mathbb{N}} \subset G'$. Consider the mapping

$$A : G \to \mathbb{C}^{\mathbb{N}}, \quad x \mapsto (\langle A_\alpha, x\rangle)_{\alpha\in\mathbb{N}}$$

In 1936, Eidelheit found a complete characterization of when this mapping is surjective for the case of Fréchet spaces. In his memory, we call such sequences $(A_\alpha)_{\alpha\in\mathbb{N}}$ Eidelheit sequences.

Given another lcHs E, we then find the mapping

$$A_E : G\varepsilon E \to E^{\mathbb{N}}, \quad L \mapsto (L(A_\alpha))_{\alpha\in\mathbb{N}}$$

We say that $(A_\alpha)_{\alpha\in\mathbb{N}}$ extends to an E-valued Eidelheit sequence if A_E is surjective.

Questions

- What does the surjectivity of A_E impose on E?
- When does the surjectivity of A imply that of A_E?

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Necessary condition for extension (1)

A lcHs E is said to satisfy factorized (A) if $E \cong \operatorname{Proj} \mathcal{E}$ where $\mathcal{E}$ is some projective spectrum consisting out of lcHs satisfying (A).

Lemma

If E is a (PLB)-space, then,

$$(\text{PA}) \Rightarrow \text{factorized (A)}$$

Corollary

- Any Fréchet space satisfies factorized (A).
- A (DF)-space satisfies factorized (A) if and only if it satisfies (A).

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- A (DF)-space satisfies factorized (A) if and only if it satisfies (A).

Necessary condition for extension (2)

Theorem

Suppose

- $G \cong k(\mathcal{A})$ for some (PLS)-sequence space $k(\mathcal{A})$ satisfying $(\mathrm{DIE})_{\theta<}$;
- E is a (PLB)-space;
- $(A_\alpha)_{\alpha\in\mathbb{N}} \subset G'$ extends to an E-valued Eidelheit sequence.

Then, E satisfies factorized (A).

The Fréchet algebra case (1)

A Fréchet algebra $(F, *)$ is called m-convex if there exists an increasing fundamental systems of seminorms $(\|\cdot\|_n)_{n\in\mathbb{N}}$ on F such that

$$\|x * y\|_n \leq \|x\|_n \|y\|_n, \qquad \forall n \in \mathbb{N}, x, y \in F$$

Theorem (Braun, 1987)

Let I be a closed ideal of a m-convex Fréchet algebra $(F, *)$ such that F/I is isomorphic, as a Fréchet algebra, to a product of finite dimensional unital commutative Banach algebras. If F satisfies (Ω), then so does I.

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The Fréchet algebra case (2)

Theorem

Let

- $(G, *)$ be a nuclear m-convex Fréchet algebra such that $G \cong k(a)$ and satisfies (Ω);
- $(A_\alpha)_{\alpha\in\mathbb{N}} \subset G'$ be an Eidelheit sequence, where each A_α is an algebra morphism;
- E be a (PLB)-space.

Consider the statements:

(I) E satisfies (PA);

(II) $(A_\alpha)_{\alpha\in\mathbb{N}}$ extends to an E-valued Eidelheit sequence;

(III) E satisfies factorized (A).

Then, (I) $\Rightarrow$ (II) $\Rightarrow$ (III).
Moreover, if E is a Fréchet space or (DF)-space, then also (III) $\Rightarrow$ (I).

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The Stieltjes moment problem

The Stieltjes moment problem asks whether one can find for any given sequence $(a_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ a non-decreasing function $\theta(t)$ on $[0,\infty)$ such that

$$\int_0^\infty t^n d\theta(t) = a_n, \qquad \forall n \in \mathbb{N}$$

- This was solved by Stieltjes in 1884.
- In 1938 Boas and Polya found that there always exists a function θ of *bounded variation*.
- In 1989 Duran showed that one may always choose θ in the Schwartz space on the real half-line.
- In 1991 he improved the result by giving a characterization of the solvability of the Stieltjes moment problem for general exponents $\{z_n\}_{n\in\mathbb{N}}$.

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The space $\mathcal{S}(\alpha, \beta; E)$

Take $-\infty \leq \beta < \alpha \leq +\infty$ and let E be any lcHs.

We define the space $S(\alpha, \beta; E)$ as all those $\varphi \in C^\infty((0, \infty); E)$ such that for any $p \in \mathrm{csn}(E)$, $k \in \mathbb{N}$, and $\beta + k + 1 < B < \alpha + k + 1$ we have

$$\sup_{t \in (0,\infty)} t^B p(\varphi^{(k)}(t)) < \infty$$

If $E = \mathbb{C}$, we simply write $\mathcal{S}(\alpha, \beta) = S(\alpha, \beta; \mathbb{C})$.

In the case where $\alpha = +\infty$ and $\beta = 0$ we find the the classical Schwartz space on the real half-line, i.e.,

$$S(\infty, 0) = \{\varphi \in \mathcal{S}(\mathbb{R}) \mid \varphi(t) = 0 \text{ if } t < 0\}$$

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Take $-\infty \leq \beta < \alpha \leq +\infty$ and let E be any lcHs.

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Solution to the general Stieltjes moment problem

Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{C}$. We put

$$\alpha = \sup\{\Re z_n \mid n \in \mathbb{N}\} \quad \text{and} \quad \beta = \inf\{\Re z_n \mid n \in \mathbb{N}\}$$

Then, we say that $(z_n)_{n\in\mathbb{N}}$ satisfies (S) if $z_n \neq z_m$ when $n \neq m$ and if either one of the following conditions holds:

(S.1) The sequence $(\Re z_n)_{n\in\mathbb{N}}$ has only one accumulation point x such that $x \neq \Re z_n$ and $x = \alpha$ or $x = \beta$;

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Theorem (Duran, 1991)

The following are equivalent:

(*a*) The sequence $(z_n)_{n\in\mathbb{N}}$ satisfies (S);

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The space $\mathcal{S}(\alpha, \beta)$ – algebraic structure

For two $\varphi, \psi \in \mathcal{S}(\alpha, \beta)$ we define the Mellin convolution as

$$\varphi *_M \psi(x) := \int_0^\infty \varphi(t)\psi\left(\frac{x}{t}\right)\frac{dt}{t}$$

Lemma

$(\mathcal{S}(\alpha, \beta), *_M)$ is an m-convex Fréchet algebra.

For any $z \in \mathbb{C}$ with $\beta < \Re z < \alpha$ the mapping

$$\mathcal{S}(\alpha, \beta) \to \mathbb{C}, \quad \varphi \mapsto \mathcal{M}\{\varphi\}(z) := \int_0^\infty t^z \varphi(t) dt$$

is well-defined and continuous, and moreover, for any $\varphi, \psi \in \mathcal{S}(\alpha, \beta)$ we have

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Theorem

$\mathcal{S}(\alpha, \beta) \cong k(a)$ with

- $a = (a_n)_{n\in\mathbb{N}} = ((1+k)^n e^{-l/n})_{n,k,l\in\mathbb{N}}$ if either α and/or β is finite;
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Corollary

$\mathcal{S}(\alpha, \beta)$ is a nuclear Fréchet space satisfying (Ω).

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The extension of the Stieltjes moment problem

Theorem

Consider the statements:

(I) $(z_n)_{n\in\mathbb{N}}$ satisfies (S) and E satisfies (PA);

(II) for every sequence $(e_n)_{n\in\mathbb{N}} \in E^{\mathbb{N}}$ there exists a function $f : \mathbb{R}_+ \to E$ such that the functions $p \circ t^{z_n} f(t) \in L^1$ for any $p \in \mathrm{csn}(E)$ and $e_n = \int_0^\infty t^{z_n} f(t)dt$ for all $n \in \mathbb{N}$;

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Then, (I) $\Rightarrow$ (II) $\Leftrightarrow$ (III) $\Rightarrow$ (IV).

Moreover, if E is a Fréchet space or (DF)-space, then also (IV) $\Rightarrow$ (I).

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