# Hermite expansions for spaces of functions with nearly optimal time-frequency decay

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# Introduction and main results

# Functions of fast time-frequency decay

Consider a function $f \in L^2(\mathbb{R})$ and its Fourier transform

$$\mathcal{F}\{f\}(\xi) = \widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-it\xi} dt.$$

In this talk, we will be interested in those functions $f$ for which the function and its Fourier transform decrease rapidly (and isotropically) at infinity.

A classical uncertainty principle by Hardy gives us a limit:

## Theorem (Hardy, 1933)

For $f \in L^2(\mathbb{R})$ suppose

$$|f(t)| \lesssim e^{-\eta t^2} \quad \text{and} \quad |\widehat{f}(\xi)| \lesssim e^{-\eta \xi^2}.$$

The following hold:

- $\eta > 1/2 : f \equiv 0.$
- $\eta = 1/2 : f(t) = ce^{-\frac{1}{2}t^2}$ for some $c \in \mathbb{C}.$

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# Functions of nearly optimal time-frequency decay

Our interest will be in functions $f \in L^2(\mathbb{R})$ with nearly optimal time-frequency decay in the following sense:

$$|f(t)| \lesssim e^{-(\frac{1}{2}-\lambda)t^2} \quad \text{and} \quad |\widehat{f}(\xi)| \lesssim e^{-(\frac{1}{2}-\lambda)\xi^2}, \qquad \forall \lambda > 0. \tag{1}$$

In particular, by [Chung, Chung, Kim, 1996] such functions belong to the Gelfand-Shilov space

$$\mathcal{S}_{\frac{1}{2}}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) \mid \sup_{m,k\in\mathbb{N}} \sup_{t\in\mathbb{R}} \frac{(1+|t|)^k |f^{(m)}(t)|}{h^m \sqrt{(k+m)!}} < \infty \text{ for some } h > 0\}$$
$$= \{f \in L^2(\mathbb{R}) \mid |f(t)| \lesssim e^{-\eta t^2} \text{ and } |\widehat{f}(\xi)| \lesssim e^{-\eta \xi^2} \text{ for some } \eta > 0\}$$

However, the space of all functions satisfying (1) does not have the same properties as $\mathcal{S}_{\frac{1}{2}}(\mathbb{R})$.
↪ E.g., it is not invariant under dilation, multiplication, and convolution.

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# Hermite functions

Consider the Hermite functions, $n \in \mathbb{N}$,

$$h_n(x) = (-1)^n \pi^{-\frac{1}{4}} (2^n n!)^{-\frac{1}{2}} e^{\frac{1}{2}x^2} \frac{d^n}{dx^n}[e^{-x^2}] = H_n(x) e^{-\frac{1}{2}x^2}$$

with $H_n$ polynomials.

They form an orthonormal basis in $L^2(\mathbb{R})$. So, for any $f \in L^2(\mathbb{R})$,

$$f = \sum_{n \in \mathbb{N}} H(f, n) h_n, \qquad \text{with} \quad H(f, n) = (f, h_n)_{L^2}.$$

Each $h_n$ is an eigenfunction of the Fourier transform $\mathcal{F}$ with eigenvalue $(-i)^n$. Therefore

$$|H(\widehat{f}, n)| = |H(f, n)|, \qquad f \in L^2(\mathbb{R}).$$

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# Hermite series expansions

The Hermite functions form an absolute Schauder basis for:

- The Schwartz space $\mathcal{S}(\mathbb{R})$ [Schwartz, 1950]
- The Gelfand-Shilov space of Roumieu type $\mathcal{S}_s(\mathbb{R})$ for $s \geq 1/2$ [Zhang, 1963]
- The Gelfand-Shilov space of Beurling type $\Sigma_s(\mathbb{R})$ for $s > 1/2$ [Pilipović, 1986]
- The general isotropic Gelfand-Shilov spaces $\mathcal{S}_{[M]}(\mathbb{R})$ [Langenbruch, 2006]

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$$\forall \eta : \quad |f(t)|, |\widehat{f}(\xi)| \lesssim (1 + |\cdot|)^{-\eta} \quad \Longleftrightarrow \quad \forall r : \quad |H(f, n)| \lesssim n^{-r}$$

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$$\exists \eta : \quad |f(t)|, |\widehat{f}(\xi)| \lesssim e^{-\eta|\cdot|^{\frac{1}{s}}} \quad \Longleftrightarrow \quad \exists r : \quad |H(f,n)| \lesssim e^{-rn^{\frac{1}{2s}}}$$

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- The general isotropic Gelfand-Shilov spaces $\mathcal{S}_{[M]}(\mathbb{R})$ [Langenbruch, 2006] :

  For a weight sequence $M = (M_n)_{n\in\mathbb{N}}$ such that $\sqrt{p!} \subseteq M_p$ (resp. $\sqrt{p!} \preceq M_p$) we have

$$\exists \eta \text{ (resp. } \forall \eta) : \quad |f(t)|, |\widehat{f}(\xi)| \lesssim e^{-\omega_M(\eta|\cdot|)}$$
$$\iff \quad \exists r \text{ (resp. } \forall r) : \quad |H(f,n)| \lesssim e^{-\omega_M(r\sqrt{n})}$$

  where

$$\omega_M(x) = \sup_{n\in\mathbb{N}} \log \frac{x^n}{M_n}.$$

# Functions with Gaussian time-frequency decay

The limit case is given by:

## Theorem (Zhang, 1963)

For any $f \in L^2(\mathbb{R})$:

$$\exists \eta > 0: \quad |f(t)|, |\widehat{f}(\xi)| \lesssim e^{-\eta|\cdot|^2} \quad \Longleftrightarrow \quad \exists r > 0: \quad |H(f,n)| \lesssim e^{-rn}.$$

Suppose that $f \in L^2(\mathbb{R})$ is such that

$$|H(f,n)| \lesssim e^{-rn}, \qquad \forall r > 0.$$

Then, it was shown that [Toft, Gumber, 2023]:

$$|f(t)| \lesssim e^{-(\frac{1}{2}-\lambda)t^2} \quad \text{and} \quad |\widehat{f}(\xi)| \lesssim e^{-(\frac{1}{2}-\lambda)\xi^2}, \qquad \forall \lambda > 0.$$

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# Functions with nearly optimal decay time-frequency decay

The goal of this talk is to show that:

## Theorem (N., Toft, Vindas, 2025)

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if and only if

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Our analysis will be based on our recent article:

L. Neyt, J. Toft, J. Vindas, *Hermite expansions for spaces of functions with nearly optimal time-frequency decay*, J. Funct. Anal. **228** (2025), article number: 110706. DOI: 10.1016/j.jfa.2024.110706.

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# Functions with specified time-frequency decay

## Theorem (N., Toft, Vindas, 2025)

Let $\omega$ be a weight function such that $\varphi(x) = \omega(e^x)$ is convex and

$$\int_x^\infty \frac{\omega(y)}{y^3}\,dx = O\left(\frac{\omega(x)}{x^2}\right), \qquad x \to \infty.$$

Then, a function $f \in L^2(\mathbb{R})$ satisfies

$$|f(t)| \lesssim e^{-\frac{1}{2}t^2+\lambda\omega(|t|)} \quad \text{and} \quad |\widehat{f}(\xi)| \lesssim e^{-\frac{1}{2}\xi^2+\lambda\omega(|\xi|)}$$

for some $\lambda > 0$ (resp. for all $\lambda > 0$) if and only if

$$|H(f,n)| \lesssim \sqrt{n!}\,e^{-\frac{1}{r}\varphi^*(rn)}$$

for some $r > 0$ (resp. for all $r > 0$) where

$$\varphi^*(x) = \sup_{y \geq 0} xy - \varphi(y).$$

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# The Fock space and the Bargmann transform

# Creation and Annihilation operators

Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\mathcal{E} = \{e_n \mid n \in \mathbb{N}\}$.

We define creation $C$ and annihilation $A$ (unbounded) operators on span $\mathcal{E}$ via ($e_{-1} \equiv 0$)

$$Ce_n = E_{\uparrow,n} \cdot e_{n+1}, \qquad Ae_n = E_{\downarrow,n} \cdot e_{n-1}, \qquad n \in \mathbb{N},$$

for certain $E_{\uparrow,n}, E_{\downarrow,n} > 0$.

- The adjoint $A = C^\dagger$ of a creation operator $C$ is an annihilation operator with $E_{\downarrow,n+1} = E_{\uparrow,n}$.
- It always holds that $[A, C] \neq 0$. For $A = C^\dagger$ we have that

$$[A, C] = I \quad \Longleftrightarrow \quad E_{\uparrow,n} = \sqrt{n+1}.$$

Suppose we are in this case. Then, for $\gamma, \theta \in \mathbb{R}$ define the operator

$$P = \gamma AC + \theta = \gamma C^\dagger C + \theta$$

which is Hermitian. Then each $e_n$ is an eigenvector of $P$ with eigenvalue $\gamma n + \theta$.

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# The Fock space

A natural example for functions on $\mathbb{C}$ is when

$$\text{creation} \;=\; \text{multiplication by } z \qquad \text{and} \qquad \text{annihilation} \;=\; \text{derivation w.r.t. } z.$$

Note that

$$\left[\frac{d}{dz}, z\right] = I.$$

Let $\mathscr{F}$ be a Hilbert space of analytic functions on $\mathbb{C}$ such that

$$(zf, g)_{\mathscr{F}} = \left(f, \frac{d}{dz}g\right)_{\mathscr{F}}, \qquad f, g \in \mathscr{F}.$$

Bargmann showed (1963) that then necessarily

$$(f, g)_{\mathscr{F}} = \frac{1}{\pi}\int_{\mathbb{C}} f(z)\overline{g(z)}e^{-|z|^2}\,dz.$$

Then $\{\frac{z^n}{\sqrt{n!}} \mid n \in \mathbb{N}\}$ forms an orthonormal basis for $\mathscr{F}$ for which $C_{\mathscr{F}} = z\cdot$ is a creation operator and $A_{\mathscr{F}} = C_{\mathscr{F}}^{\dagger} = \frac{d}{dz}$ an annihilation operator.

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$$(f, g)_{\mathscr{F}} = \frac{1}{\pi}\int_{\mathbb{C}} f(z)\overline{g(z)}e^{-|z|^2}\,dz.$$

Then $\{\frac{z^n}{\sqrt{n!}} \mid n \in \mathbb{N}\}$ forms an orthonormal basis for $\mathscr{F}$ for which $C_{\mathscr{F}} = z\cdot$ is a creation operator and $A_{\mathscr{F}} = C_{\mathscr{F}}^{\dagger} = \frac{d}{dz}$ an annihilation operator.

# The Fock space

A natural example for functions on $\mathbb{C}$ is when

$$\text{creation} = \text{multiplication by } z \quad \text{and} \quad \text{annihilation} = \text{derivation w.r.t. } z.$$

Note that

$$\left[\frac{d}{dz}, z\right] = I.$$

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# The abstract Bargmann transform

We wish to find a unitary mapping $\mathfrak{B}_{\mathcal{E}} : \mathcal{H} \to \mathscr{F}$ such that

$$\mathfrak{B}_{\mathcal{E}}[Cf](z) = z\mathfrak{B}_{\mathcal{E}}[f](z), \qquad \mathfrak{B}_{\mathcal{E}}[Af](z) = \frac{d}{dz}\mathfrak{B}_{\mathcal{E}}[f](z), \qquad f \in \operatorname{span} \mathcal{E}.$$

This is possible if and only if $A = C^\dagger$ and $[A, C] = I$. In this case, we necessarily have

$$\mathfrak{B}_{\mathcal{E}}[e_n](z) = \frac{z^n}{\sqrt{n!}}.$$

As the Fock space allows continuous evaluation at points, there exists by the Riesz representation theorem an anti-holomorphic mapping

$$K : \mathbb{C} \to \mathcal{H}, \quad z \mapsto K(z).$$

such that

$$\mathfrak{B}_{\mathcal{E}}[f](z) = (f, K(z))_{\mathcal{H}}, \qquad f \in \mathcal{H}.$$

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# The Hermite functions

Consider the operators

$$C = \frac{1}{\sqrt{2}}\left(t - \frac{d}{dt}\right), \qquad A = \frac{1}{\sqrt{2}}\left(t + \frac{d}{dt}\right).$$

Then for the Hermite functions $h_n$ we have $(h_{-1} \equiv 0)$

$$Ch_n = \sqrt{n+1}h_{n+1}, \qquad Ah_n = \sqrt{n}h_{n-1}, \qquad n \in \mathbb{N}.$$

In particular, $A = C^\dagger$ and $[A, C] = I$.

Note that

$$H = AC + \frac{1}{2} = \frac{t^2 - \Delta}{2}$$

is the quantum harmonic oscillator.

# The Bargmann transform

We find the system of partial differential equations

$$\frac{d}{dt}K(z,t) = (\sqrt{2}\overline{z} - t)K(z,t), \qquad \frac{d}{d\overline{z}}K(z,t) = \frac{1}{\sqrt{2}}\left(t - \frac{d}{dt}\right)K(z,t)$$

Solving this gives us the Bargmann transform

$$\mathfrak{B}[f](z) = \pi^{-\frac{1}{4}} \int_{-\infty}^{\infty} f(t) e^{-\frac{1}{2}(z^2+t^2)+\sqrt{2}tz}\, dt, \qquad f \in L^2(\mathbb{R}).$$

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# The Pilipović spaces

# Weight functions

A weight function $\omega$ is an unbounded non-decreasing function $[0,\infty) \to [0,\infty)$.

A weight function $\omega$ satisfies $(\alpha)$ if for some $L \geq 1$:

$$\omega(x+y) \leq L[\omega(x)+\omega(y)+1], \qquad x,y \geq 0,$$

or equivalently

$$\sup_{x\geq 0} \frac{\omega(2x)}{\omega(x)} < \infty.$$

A weight function $\omega$ satisfies $(\delta)$ if the function $\varphi(x) = \omega(e^x)$ is convex.
In this case, we may consider the *Young conjugate* $\varphi^*$ of $\varphi$

$$\varphi^*(x) = \sup_{y\geq 0} xy - \varphi(y).$$

Then, $\varphi^*$ is convex, unbounded, and non-decreasing such that $(\varphi^*)^* = \varphi$.
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# Generalized Pilipović spaces

Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\mathcal{E} = \{e_n \mid n \in \mathbb{N}\}$. We consider the mapping

$$H : \mathcal{H} \times \mathbb{N} \to \mathbb{C}, \quad f \mapsto H(f, n) = (f, e_n)_{\mathcal{H}}.$$

Let $\omega$ be a weight function satisfying $(\delta)$. For any $r > 0$ we define the Banach space

$$\mathcal{H}_{\omega,r} = \left\{ f \in \mathcal{H} \mid \sup_{n \in \mathbb{N}} \frac{|H(f,n)|}{\sqrt{n!}} e^{\frac{1}{r}\varphi^*(rn)} < \infty \right\}.$$

Then, we put

$$\mathcal{H}_\omega = \bigcup_{r>0} \mathcal{H}_{\omega,r}, \qquad \mathcal{H}_{0,\omega} = \bigcap_{r>0} \mathcal{H}_{\omega,r},$$

i.e. for $f \in \mathcal{H}$

$$f \in \mathcal{H}_\omega \ (\text{resp. } f \in \mathcal{H}_{0,\omega})$$

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# Generalized Pilipović spaces – invariance under $\ell^p$-norm

## Lemma

For any $p \in [1, \infty)$ a function $f \in \mathcal{H}$ belongs to $\mathcal{H}_\omega$ (resp. $\mathcal{H}_{0,\omega}$) if and only if for some $r > 0$ (resp. for all $r > 0$)

$$\sum_{n \in \mathbb{N}} \left( \frac{|H(f, n)|}{\sqrt{n!}} e^{\frac{1}{r}\varphi^*(rn)} \right)^p < \infty$$

# Examples of Generalized Pilipović spaces

The following are classical examples of Pilipović spaces:

| $\mathcal{H}_\omega$ | $\omega(x)$ | $H(f,n) \lesssim$ | parameter |
|---|---|---|---|
| $\mathcal{H}_0$ | $\log_+(x)$ | $\sqrt{n!} \cdot 1_{[0,1]}(rn)$ | |
| $\mathcal{H}_s$ | $\log(1+x)^{\frac{1}{1-2s}}$ | $e^{-rn^{\frac{1}{2s}}}$ | $0 < s < 1/2$ |
| $\mathcal{H}_{\flat_\sigma}$ | $x^{\frac{2\sigma}{\sigma+1}}$ | $r^n(n!)^{-\frac{1}{2\sigma}}$ | $\sigma > 0$ |
| $\mathcal{H}_{0,\frac{1}{2}}$ | $x^2$ | $e^{-rn}$ | |

Note that in particular

$$\mathcal{H}_{0,0} = \langle e_0 \rangle, \qquad \mathcal{H}_0 = \langle \mathcal{E} \rangle.$$

In general, when $\log x = o(\omega(x))$ and $\omega(x) = o(x^2)$, we have

$$\mathcal{H}_{0,0} \subset \mathcal{H}_0 \hookrightarrow \mathcal{H}_{0,\omega} \hookrightarrow \mathcal{H}_\omega \hookrightarrow \mathcal{H}_{0,\frac{1}{2}} \hookrightarrow \mathcal{H}.$$

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# Bargmann transform of the generalized Pilipović spaces

## Proposition

Let $\omega$ be a weight function satisfying $(\delta)$.

For any entire function $F$ there exists a $f \in \mathcal{H}_\omega$ (resp. $f \in \mathcal{H}_{0,\omega}$) such that

$$F = \mathfrak{B}_\mathcal{E}[f]$$

if and only if

$$|F(z)| \lesssim e^{\lambda\omega(|z|)}$$

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From now on, we let $\mathcal{H} = L^2(\mathbb{R})$ and $\mathcal{E} = \{h_n \mid n \in \mathbb{N}\}$, the Hermite functions.

When $\log x = o(\omega(x))$ and $\omega(x) = o(x^2)$, we have

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Recall the Bargmann transform is given here by

$$\begin{aligned}\mathfrak{B}[f](z) &= \pi^{-\frac{1}{4}} \int_{-\infty}^{\infty} f(t) e^{-\frac{1}{2}(z^2+t^2)+\sqrt{2}tz}\, dt \\ &= \left(\pi^{-\frac{1}{4}} \int_{-\infty}^{\infty} \left[f(t)e^{-\frac{1}{2}t^2}\right] e^{\sqrt{2}tz}\, dt\right) \cdot e^{-\frac{1}{2}z^2}.\end{aligned}$$

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# The short-time Fourier transform

Let $\phi(t) = h_0(t) = \pi^{-\frac{1}{4}} e^{-\frac{1}{2}t^2}$ be the (normalized) Gaussian.

The short-time Fourier transform (w.r.t. $\phi$) of a function $f \in L^2(\mathbb{R})$ is given by

$$V_\phi f(t,\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\phi(x-t)e^{-ix\xi}dt = \mathcal{F}\{f \cdot \phi(\cdot - t)\}(\xi).$$

It gives a unitary operator $V_\phi : L^2(\mathbb{R}) \to L^2(\mathbb{R}^2)$.

Observe that:

- By the Plancherel identity

$$V_\phi f(t,\xi) = e^{-it\xi} V_\phi \widehat{f}(\xi, -t)$$

- By the Fourier inversion formula

$$\begin{aligned} f\left(\frac{t}{2}\right) e^{-\frac{1}{8}t^2} &= \pi^{\frac{1}{4}} f\left(\frac{t}{2}\right) \phi\left(\frac{t}{2} - t\right) \\ &= \pi^{\frac{1}{4}} \mathcal{F}^{-1}\left\{\mathcal{F}\left\{f \cdot \phi(\cdot - t)\right\}\right\}\left(\frac{t}{2}\right) \\ &= \pi^{\frac{1}{4}} \int_{-\infty}^{\infty} V_\phi f(t,\xi) e^{\frac{i}{2}t\xi} d\xi \end{aligned}$$

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$$V_\phi f(t,\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\phi(x-t)e^{-ix\xi}dt = \mathcal{F}\{f \cdot \phi(\cdot - t)\}(\xi).$$

It gives a unitary operator $V_\phi : L^2(\mathbb{R}) \to L^2(\mathbb{R}^2)$.

Observe that:

- By the Plancherel identity

$$V_\phi f(t,\xi) = e^{-it\xi} V_\phi \widehat{f}(\xi, -t)$$

- By the Fourier inversion formula

$$\begin{aligned} f\left(\frac{t}{2}\right) e^{-\frac{1}{8}t^2} &= \pi^{\frac{1}{4}} f\left(\frac{t}{2}\right) \phi\left(\frac{t}{2} - t\right) \\ &= \pi^{\frac{1}{4}} \mathcal{F}^{-1}\left\{\mathcal{F}\left\{f \cdot \phi(\cdot - t)\right\}\right\}\left(\frac{t}{2}\right) \\ &= \pi^{\frac{1}{4}} \int_{-\infty}^{\infty} V_\phi f(t,\xi) e^{\frac{i}{2}t\xi} d\xi \end{aligned}$$

# The short-time Fourier transform

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# Bargmann transform vs. short-time Fourier transform

For any $f \in L^2(\mathbb{R})$ and $z = t + i\xi$ we find

$$\begin{aligned}
\mathfrak{B}[f](z) &= \left(\pi^{-\frac{1}{4}} \int_{-\infty}^{\infty} \left[f(x)e^{-\frac{1}{2}x^2}\right] e^{\sqrt{2}xz}\, dx\right) \cdot e^{-\frac{1}{2}z^2} \\
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Consequently, bounds on the Bargmann transform correspond to time-frequency decay.

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# The Bargmann transform and time-frequency decay

## Lemma

Let $\omega$ be a weight function satisfying $(\alpha)$ s.t. $\omega(x) = o(x^2)$ (resp. $\omega(x) = O(x^2)$).

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# An optimal Phragmén-Lindelöf type result on sectors

# A classical Phragmén-Lindelöf type result

For any $\varrho > 0$ we write

$$S_\varrho = \{z \in \mathbb{C} \mid -\varrho/2 < \arg z < \varrho/2\}.$$

**Lemma (Classical Phragmén-Lindelöf principle on sectors)**

Let $\sigma > 0$.

Suppose $F$ is analytic in $S_{\pi/\sigma}$ with continuous extension to $\overline{S}_{\pi/\sigma}$ such that

$$|F(z)| \leq M, \qquad z \in \partial S_{\pi/\sigma},$$

and, for all $\varepsilon > 0$,

$$|F(z)| \lesssim_\varepsilon e^{\varepsilon |z|^\sigma}, \qquad z \in S_{\pi/\sigma}.$$

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# An optimal result

## Theorem (N., Toft, Vindas, 2025)

Let $\sigma > 0$ and $\omega$ be a weight function satisfying $(\alpha)$ and $\int_1^\infty \frac{\omega(s)}{s^{\sigma+1}} ds < \infty$.

The following statements are equivalent:

(I) $\int_1^\infty \frac{\omega(xs)}{s^{\sigma+1}} ds = O(\omega(x))$.

(II) There exists $A > 0$ such that for any $\lambda > 0$:
There exists $B > 0$ such that for any $F$ analytic in $S_{\pi/\sigma}$ with continuous extension to $\overline{S}_{\pi/\sigma}$ for which

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# Sketch of the proof

Using the Poisson transform

$$P_\omega(x,y) = \frac{x}{\pi}\int_{-\infty}^{\infty} \frac{\omega(t^{\frac{1}{\sigma}})}{x^2+(y-t)^2}\,dt,$$

one constructs $F_\omega$ that is analytic on $S_{\pi/\sigma}$ with continuous extension to $\overline{S}_{\pi/\sigma}$ satisfying the following properties:

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**(I)** $\Rightarrow$ **(II)** : Apply the classical Phragmén-Lindelöf result to $G = F \cdot F_\omega^{-\lambda/\nu_1}$.

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**(I)** $\Rightarrow$ **(II)** : Apply the classical Phragmén-Lindelöf result to $G = F \cdot F_\omega^{-\lambda/\nu_1}$.

**(II)** $\Rightarrow$ **(I)** : Apply **(II)** to $F_\omega$.

# Sketch of the proof

Using the Poisson transform

$$P_\omega(x,y) = \frac{x}{\pi}\int_{-\infty}^{\infty} \frac{\omega(t^{\frac{1}{\sigma}})}{x^2+(y-t)^2}\,dt,$$

one constructs $F_\omega$ that is analytic on $S_{\pi/\sigma}$ with continuous extension to $\overline{S}_{\pi/\sigma}$ satisfying the following properties:

(i) For some $c, \nu_1 > 0$

$$\forall \varepsilon > 0: \quad ce^{\nu_1\omega(|z|)} \leq |F_\omega(z)| \lesssim e^{\varepsilon|z|^\sigma}, \qquad z \in S_{\pi/\sigma}.$$

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# Proofs of main results – case of $\omega(x) = x^2$

Suppose we have, for all $\lambda > 0$,

$$|\mathfrak{B}[f](z)| \lesssim e^{\frac{1}{2}t^2+\lambda\xi^2} \quad \text{and} \quad |\mathfrak{B}[f](z)| \lesssim e^{\frac{1}{2}\xi^2+\lambda t^2}, \qquad z = t + i\xi.$$

Fix some $\varepsilon \in (0, 1)$ and let $\vartheta = \arctan(1/4\varepsilon)$. Put

$$G_\varepsilon(z) = \mathfrak{B}[f](z)e^{-\varepsilon(ze^{-i\frac{\pi}{4}})^2}.$$

Then, for all $\lambda > 0$,

$$|G_\varepsilon(r)| \lesssim e^{\lambda r^2} \quad \text{and} \quad |G_\varepsilon(re^{i\vartheta})| \lesssim e^{\lambda r^2}, \qquad r > 0.$$

By the Phragmén-Lindelöf principle we then have

$$|\mathfrak{B}[f](z)| \lesssim e^{2\varepsilon|z|^2}, \qquad 0 \le \arg z \le \vartheta < \frac{\pi}{2}.$$

Finally, for any $\varrho \in (\vartheta, \pi/2]$ we have

$$|\mathfrak{B}[f](re^{i\varrho})| \lesssim e^{\frac{1}{2}\cos^2\vartheta\cdot r^2+\varepsilon r^2} \le e^{9\varepsilon r^2}, \qquad r > 0.$$

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# Proofs of main results – case when $\int_x^\infty \frac{\omega(x)}{x^3}dx = O\left(\frac{\omega(x)}{x^2}\right)$

Suppose we have, for some $\lambda > 0$,

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Since

$$\frac{\omega(x)}{x^2} = 2\int_x^\infty \frac{\omega(s)}{s^3}ds = o(1),$$

it follows from the previous result that, for every $\varepsilon > 0$,

$$|\mathfrak{B}[f](z)| \lesssim_\varepsilon e^{\varepsilon|z|^2}.$$

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# The multidimensional case and future perspectives

# The $d$-dimensional case

Let $\mathcal{F}$ denote the one-dimensional Fourier transform. We write

$$\mathcal{F}_\varpi = \bigotimes_{j=1}^{d} \mathcal{F}^{\varpi_j}, \qquad \varpi \in \{0,1\}^d.$$

Then $\mathcal{F}_0 = \mathrm{id}$ while $\mathcal{F}_{\mathbf{1}} = \mathcal{F}_{(1,\dots,1)}$ is the $d$-dimensional Fourier transform.

The $d$-dimensional Hermite functions

$$h_\alpha(x) = (-1)^{|\alpha|} \pi^{-\frac{d}{4}} (2^{|\alpha|}\alpha!)^{-\frac{1}{2}} e^{\frac{1}{2}|x|^2} \frac{\partial^\alpha}{\partial x^\alpha}[e^{-|x|^2}], \qquad \alpha \in \mathbb{N}^d,$$

form an orthonormal basis for $L^2(\mathbb{R}^d)$.

We write for $f \in L^2(\mathbb{R}^d)$

$$f = \sum_{\alpha \in \mathbb{N}^d} H(f,\alpha) h_\alpha, \qquad \text{with } H(f,\alpha) = \langle f, h_\alpha \rangle .$$

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# Multi-dimensional analogs of the main results (1)

## Theorem (N., Toft, Vindas, 2025)

A function $f \in L^2(\mathbb{R}^d)$ satisfies

$$|\mathcal{F}_\varpi\{f\}(\xi)| \lesssim e^{-(\frac{1}{2}-\lambda)|\xi|^2}, \qquad \varpi \in \{0,1\}^d, \xi \in \mathbb{R}^d,$$

for all $\lambda > 0$, if and only if

$$|H(f,\alpha)| \lesssim e^{-r|\alpha|}, \qquad \alpha \in \mathbb{N}^d,$$

for all $r > 0$.

# Multi-dimensional analogs of the main results (2)

## Theorem (N., Toft, Vindas, 2025)

Let $\omega$ be a weight function such that $\varphi(x) = \omega(e^x)$ is convex and

$$\int_x^\infty \frac{\omega(x)}{x^3}dx = O\left(\frac{\omega(x)}{x^2}\right), \qquad x \to \infty.$$

Then, a function $f \in L^2(\mathbb{R}^d)$ satisfies

$$|\mathcal{F}_\varpi\{f\}(\xi)| \lesssim e^{-\frac{1}{2}|\xi|^2+\lambda\omega(|\xi|)}, \qquad \varpi \in \{0,1\}^d, \xi \in \mathbb{R}^d,$$

for some $\lambda > 0$ (resp. for all $\lambda > 0$) if and only if

$$|H(f,\alpha)| \lesssim \sqrt{n!}e^{-\frac{1}{r}\varphi^*(r|\alpha|)}, \qquad \alpha \in \mathbb{N}^d,$$

for some $r > 0$ (resp. for all $r > 0$).

# Some open problems

- In the multi-dimensional case, does one necessarily need the decay of all partial Fourier transforms?
  $\hookrightarrow$ For $\mathcal{H}_{0,0}$ and $\mathcal{H}_0$ just $\mathcal{F}_0$ and $\mathcal{F}_1$ suffice [Bonami, Demange, Jaming, 2003].

- Can we characterize nearly optimal (isotropic) time-frequency decay using other orthonormal bases of $L^2(\mathbb{R}^d)$?
  $\hookrightarrow$ Let $P$ be a normal globally elliptic Shubin-type differential operator with orthonormal basis $\{u_j : j \in \mathbb{N}\}$ of eigenfunctions of $P$. Then, for $f \in L^2(\mathbb{R}^d)$, [Vučković, Vindas, 2016]:

$$f \in \mathcal{S}_{\frac{1}{2}}(\mathbb{R}^d) \iff \exists r > 0 : \ |\langle f, u_j \rangle| \lesssim e^{-r j^{\frac{1}{d}}}.$$

- Can we characterize nearly optimal anisotropic time-frequency decay using orthonormal bases of $L^2(\mathbb{R}^d)$?
  $\hookrightarrow$ The elements of $\mathcal{S}_q^p(\mathbb{R}^d)$ $(p, q \in \mathbb{Q},\ p + q = 1)$ are characterized via their expansions through orthonormal bases coming from anisotropic Shubin operators [Cappiello, Gramchev, Pilipović, Rodino, 2019].

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# Thank you for your attention!

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