Quantitative uncertainty principles for time-frequency Gaussian decay

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Hardy's uncertainty principle

For a function $f \in L^1(\mathbb{R}^d)$ we consider its Fourier transform

$$\mathcal{F}\{f\}(\xi) = \widehat{f}(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(t) e^{-i\langle t,\xi\rangle}\, dt.$$

The classical uncertainty principle by Hardy shows that the Gaussian has optimal isotropic time-frequency decay:

Theorem (Hardy, 1933)

Let $f \in L^2(\mathbb{R}^d)$ be such that for some $\eta > 0$

$$|f(x)| \lesssim e^{-\eta|x|^2} \quad \text{and} \quad |\widehat{f}(\xi)| \lesssim e^{-\eta|\xi|^2}.$$

Then:

- $\eta > 1/2 \implies f \equiv 0$;
- $\eta = 1/2 \implies f(x) = ce^{-\frac{1}{2}|x|^2}$ for some $c \in \mathbb{C}$.

Hardy's uncertainty principle – Quantified

Hardy quantified further:

Theorem (Hardy, 1933)

For any $f \in L^2(\mathbb{R}^d)$: The time-frequency decay

$$|f(x)| \lesssim (1+|x|)^N e^{-\frac{1}{2}|x|^2} \quad \text{and} \quad |\widehat{f}(\xi)| \lesssim (1+|\xi|)^N e^{-\frac{1}{2}|\xi|^2}$$

holds for some $N \in \mathbb{N}$ if and only if

$$f(x) = P(x)e^{-\frac{1}{2}|x|^2}$$

for some polynomial P of degree at most N.

Hardy's uncertainty principle – Quantified

In fact, coordinate-wise estimates suffice:

Theorem (Bonami, Demange, Jaming, 2003)

For any $f \in L^2(\mathbb{R}^d)$: For $j \in \{1, \dots, d\}$, the time-frequency decay

$$|f(x)| \lesssim (1 + |x_j|)^N e^{-\frac{1}{2}|x_j|^2} \quad \text{and} \quad |\widehat{f}(\xi)| \lesssim (1 + |\xi_j|)^N e^{-\frac{1}{2}|\xi_j|^2}$$

holds for some $N \in \mathbb{N}$ if and only if

$$f(x) = P(x) e^{-\frac{1}{2}|x|^2}$$

for some polynomial P of degree at most N.

Question: Can we quantify slower time-frequency decay?

Hermite coefficients

Consider the Hermite functions

$$h_\alpha(x) = (-1)^{|\alpha|}\pi^{-\frac{d}{4}}(2^{|\alpha|}\alpha!)^{-\frac{1}{2}}e^{\frac{1}{2}|x|^2}\frac{\partial^\alpha}{\partial x^\alpha}[e^{-|x|^2}], \qquad \alpha \in \mathbb{N}^d.$$

They form an orthonormal basis in $L^2(\mathbb{R}^d)$. So, for any $f \in L^2(\mathbb{R}^d)$,

$$f = \sum_{\alpha\in\mathbb{N}^d} H(f,\alpha)h_\alpha, \qquad \text{with} \quad H(f,\alpha) = (f, h_\alpha)_{L^2}.$$

They are eigenfunctions of the Fourier transform:

$$\widehat{h_\alpha} = (-i)^\alpha h_\alpha, \qquad \alpha \in \mathbb{N}^d.$$

We find the unitary map

$$L^2(\mathbb{R}^d) \to \ell^2(\mathbb{N}^d), \quad f \mapsto (H(f,\alpha))_{\alpha\in\mathbb{N}^d}.$$

Note that

$$|H(\widehat{f},\alpha)| = |H(f,\alpha)|, \qquad f \in L^2(\mathbb{R}^d).$$

Time-frequency decay vs. Hermite series expansions

The Hermite functions form an absolute Schauder basis for:

- The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ [Schwartz, 1950]
- The Gelfand-Shilov space of Roumieu type $\mathcal{S}_s(\mathbb{R}^d)$, $s \geq 1/2$, [Zhang, 1963]
- The Gelfand-Shilov space of Beurling type $\Sigma_s(\mathbb{R}^d)$, $s > 1/2$, [Pilipović, 1986]
- The general isotropic Gelfand-Shilov spaces $\mathcal{S}_{[M]}(\mathbb{R}^d)$ [Langenbruch, 2006]

Time-frequency decay vs. Hermite series expansions

The Hermite functions form an absolute Schauder basis for:

- The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ [Schwartz, 1950] :

$$\forall \eta : \quad |f(x)|, |\widehat{f}(\xi)| \lesssim (1 + |\cdot|)^{-\eta} \quad \Longleftrightarrow \quad \forall r : \quad |H(f, \alpha)| \lesssim |\alpha|^{-r}$$

- The Gelfand-Shilov space of Roumieu type $\mathcal{S}_s(\mathbb{R}^d)$, $s \geq 1/2$, [Zhang, 1963]
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$$\exists \eta : \quad |f(x)|, |\widehat{f}(\xi)| \lesssim e^{-\eta|\cdot|^{\frac{1}{s}}} \quad \Longleftrightarrow \quad \exists r : \quad |H(f,\alpha)| \lesssim e^{-r|\alpha|^{\frac{1}{2s}}}$$

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$$\forall \eta: \quad |f(x)|, |\widehat{f}(\xi)| \lesssim e^{-\eta|\cdot|^{\frac{1}{s}}} \quad \Longleftrightarrow \quad \forall r: \quad |H(f,\alpha)| \lesssim e^{-r|\alpha|^{\frac{1}{2s}}}$$

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  For a weight sequence $M = (M_n)_{n\in\mathbb{N}}$ such that $\sqrt{p!} \subseteq M_p$ (resp. $\sqrt{p!} \preceq M_p$) we have

$$\exists \eta \text{ (resp. } \forall \eta) : \quad |f(x)|, |\widehat{f}(\xi)| \lesssim e^{-\omega_M(\eta|\cdot|)}$$

$$\iff \quad \exists r \text{ (resp. } \forall r) : \quad |H(f,\alpha)| \lesssim e^{-\omega_M(r\sqrt{|\alpha|})}$$

  where

$$\omega_M(u) = \sup_{n\in\mathbb{N}} \log \frac{u^n}{M_n}.$$

Differentiability

The previous function spaces may all be characterized using differentiability conditions.

For example, in the minimal case, we have by [Chung, Chung, Kim, 1996] that $f \in L^2(\mathbb{R}^d)$ satisfies

$$|f(x)| \lesssim e^{-\eta|x|^2}, \qquad |\widehat{f}(\xi)| \lesssim e^{-\eta|\xi|^2}$$

for some $0 < \eta \leq 1/2$, if and only if, for some $h > 0$,

$$\sup_{\alpha,\beta\in\mathbb{N}^d} \sup_{x\in\mathbb{R}^d} \frac{|x^\beta f^{(\alpha)}(t)|}{h^{\alpha+\beta}\sqrt{(\alpha+\beta)!}} < \infty.$$

By Zhang's result, we have that it is equivalent to

$$|H(f,\alpha)| \lesssim \varepsilon^{|\alpha|}$$

for some $0 < \varepsilon < 1$.

Recent results

Can we close the gap between Hardy's (resp. Bonami, Demange, and Jaming their) result and that of Zhang?

- In

  J. Toft, A. Gumber, *Fourier characterizations of Pilipović spaces*, J. Func. Anal. **284** (2023), 109724

  it was shown that specified bounds on the Hermite coefficients lead to certain time-frequency decay. But one needed decay estimates on the *Fractional Fourier transform* to go back.

- For dimension 1, we showed that decay estimates on the function and its Fourier transform suffice in

  L. Neyt, J. Toft, J. Vindas, *Hermite expansions for spaces of functions with nearly optimal time-frequency decay*, J. Funct. Anal. **288** (2025), 110706.

- Finally, the multidimensional case was covered in our recent preprint

  L. Neyt, J. Toft, J. Vindas, *Quantitative uncertainty principles for time-frequency Gaussian decay*, arXiv: 2508.03273.

The Bargmann transform

The Bargmann transform gives a unitary map

$$\mathcal{B}_d : L^2(\mathbb{R}^d) \to \mathscr{F}_2, \quad f \mapsto \left(z \mapsto \mathcal{B}_d f(z) = \sum_{\alpha \in \mathbb{N}^d} \frac{H(f,\alpha)}{\sqrt{\alpha!}} z^\alpha \right)$$

with

$$\mathscr{F}_2 = \{F \in \mathcal{A}(\mathbb{C}^d) \mid \int_{\mathbb{C}^d} |F(z)|^2 e^{-|z|^2} dz < \infty\}$$

the Fock space.

Note that

$$\mathcal{B}_d h_\alpha = \frac{1}{\sqrt{\alpha!}} z^\alpha$$

Explicitly, the integral transform is given by

$$\mathcal{B}_d f(z) = \pi^{-\frac{d}{4}} \int_{\mathbb{R}^d} f(t) e^{-\frac{1}{2}\left(\langle z,z\rangle + |t|^2\right) + \sqrt{2}\langle z,t\rangle} dt.$$

Weight functions

A weight function ω is an unbounded non-decreasing function $[0,\infty) \to [0,\infty)$.

A weight function ω satisfies (α) if for some $L \geq 1$:

$$\omega(x+y) \leq L[\omega(x)+\omega(y)+1], \qquad x,y \geq 0,$$

or equivalently

$$\sup_{x\geq 0} \frac{\omega(2x)}{\omega(x)} < \infty.$$

A weight function ω satisfies (δ) if the function $\varphi(x) = \omega(e^x)$ is convex.
In this case, we may consider the *Young conjugate* φ^* of φ

$$\varphi^*(x) = \sup_{y\geq 0} xy - \varphi(y).$$

Then, φ^* is convex, unbounded, and non-decreasing such that $(\varphi^*)^* = \varphi$.
Moreover, either $\omega \asymp \log^+$ or $\log x = o(\omega(x))$.

Bargmann transform for specified Hermite coefficient bounds

Proposition

Let ω be a weight function satisfying (δ).

For any entire function F there exists a $f \in L^2(\mathbb{R}^d)$ such that

$$|H(f,\alpha) \lesssim \sqrt{\alpha!} e^{-\frac{1}{r}\varphi^*(r|\alpha|)}$$

for some $r > 0$ (resp. for all $r > 0$) and

$$F = \mathfrak{B}_d f$$

if and only if

$$|F(z)| \lesssim e^{\lambda\omega(|z|)}$$

for some $\lambda > 0$ (resp. for all $\lambda > 0$).

The Bargmann transform and time-frequency decay

Lemma

Let ω be a weight function satisfying (α) s.t. $\omega(x) = o(x^2)$ (resp. $\omega(x) = O(x^2)$).

For a function $f \in L^2(\mathbb{R}^d)$, consider the statements:

(i) $H(f,\alpha) \lesssim \sqrt{\alpha!} e^{-\frac{1}{r}\varphi^*(r|\alpha|)}$ for some $r > 0$ (resp. for all $r > 0$).

(ii) $|\mathfrak{B}_d f(z)| \lesssim e^{\lambda\omega(|z|)}$ for some $\lambda > 0$ (resp. for all $\lambda > 0$).

(iii) For some $\lambda > 0$ (resp. for all $\lambda > 0$)

$$|f(t)| \lesssim e^{-\frac{1}{2}|t|^2+\lambda\omega(|t|)} \quad \text{and} \quad |\widehat{f}(\xi)| \lesssim e^{-\frac{1}{2}|\xi|^2+\lambda\omega(|\xi|)}.$$

(iv) For some $\lambda > 0$ (resp. for all $\lambda > 0$)

$$|\mathfrak{B}_d f(z)| \lesssim e^{\frac{1}{2}|t|^2+\lambda\omega(|\xi|)} \quad \text{and} \quad |\mathfrak{B}_d f(z)| \lesssim e^{\frac{1}{2}|\xi|^2+\lambda\omega(|t|)}, \quad z = t+i\xi \in \mathbb{C}^d$$

$\hookrightarrow$ In particular, $|\mathcal{B}_d f(\cdot)| \lesssim e^{\lambda\omega(|\cdot|)}$ on $\mathbb{R}^d \cup i\,\mathbb{R}^d$.

Then, (i) $\iff$ (ii) $\implies$ (iii) $\implies$ (iv).

A classical Phragmén-Lindelöf type result

For any $\varrho > 0$ we write

$$S_\varrho = \{z \in \mathbb{C} \mid -\varrho/2 < \arg z < \varrho/2\}.$$

Note: We are working in dimension one.

Lemma (Classical Phragmén-Lindelöf principle on sectors)

Let $\sigma > 0$.

Suppose F is analytic in $S_{\pi/\sigma}$ with continuous extension to $\overline{S}_{\pi/\sigma}$ such that

$$|F(z)| \leq M, \qquad z \in \partial S_{\pi/\sigma},$$

and, for all $\varepsilon > 0$,

$$|F(z)| \lesssim_\varepsilon e^{\varepsilon |z|^\sigma}, \qquad z \in S_{\pi/\sigma}.$$

Then,

$$|F(z)| \leq M, \qquad z \in S_{\pi/\sigma}.$$

An optimal result

Theorem (N., Toft, Vindas, 2025)

Let $\sigma > 0$ and ω be a weight function satisfying (α) and $\int_1^\infty \frac{\omega(s)}{s^{\sigma+1}} ds < \infty$.

The following statements are equivalent:

(I) $\int_1^\infty \frac{\omega(xs)}{s^{\sigma+1}} ds = O(\omega(x))$.

(II) There exists $A > 0$ such that for any $\lambda > 0$:
There exists $B > 0$ such that for any F analytic in $S_{\pi/\sigma}$ with continuous extension to $\overline{S}_{\pi/\sigma}$ for which

$$|F(z)| \leq Me^{\lambda\omega(|z|)}, \qquad z \in \partial S_{\pi/\sigma},$$

and, for all $\varepsilon > 0$,

$$|F(z)| \lesssim_\varepsilon e^{\varepsilon|z|^\sigma}, \qquad z \in S_{\pi/\sigma},$$

then

$$|F(z)| \leq BMe^{A\lambda\omega(|z|)}, \qquad z \in S_{\pi/\sigma}.$$

Key to showing the multidimensional case

Proposition (N., Toft, Vindas)

Let ω be a weight function satisfying (α) and (δ).

Let F be an entire function on $\mathbb{C}^d$ such that for some $\lambda > 0$ (for all $\lambda > 0$)

$$\sup_{u \in \mathbb{S}^{d-1}} |F_u(z)| = \sup_{u \in \mathbb{S}^{d-1}} |F(u_1 z, \dots, u_d z)| \lesssim e^{\lambda \omega(z)}, \qquad z \in \mathbb{C},$$

where $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d \mid |x| = 1\}$.

Then

$$F^{(\alpha)}(0) \lesssim \alpha! e^{-\frac{1}{r}\varphi^*(r|\alpha|)}$$

for some $r > 0$ (for all $r > 0$).

Main result – $\omega(t) = t^2$

Theorem (N., Toft, Vindas, $d = 1$ – 2025; $d \geq 2$ preprint)

For any $f \in L^2(\mathbb{R}^d)$:

$$|f(t)| \lesssim e^{-(\frac{1}{2}-\lambda)|t|^2} \quad \text{and} \quad |\widehat{f}(\xi)| \lesssim e^{-(\frac{1}{2}-\lambda)|\xi|^2}, \qquad \forall \lambda > 0,$$

if and only if

$$|H(f,\alpha)| \lesssim \varepsilon^{|\alpha|}, \qquad \forall \varepsilon > 0.$$

Main result – $\omega(t) = o(t^2)$

Theorem (N., Toft, Vindas, $d = 1$ – 2025; $d \geq 2$ preprint)

Let ω be a weight function such that $\varphi(x) = \omega(e^x)$ is convex and

$$\int_x^\infty \frac{\omega(y)}{y^3} dy = O\left(\frac{\omega(x)}{x^2}\right), \qquad x \to \infty.$$

Then, a function $f \in L^2(\mathbb{R}^d)$ satisfies

$$|f(t)| \lesssim e^{-\frac{1}{2}|t|^2 + \lambda\omega(|t|)} \quad \text{and} \quad |\widehat{f}(\xi)| \lesssim e^{-\frac{1}{2}|\xi|^2 + \lambda\omega(|\xi|)}$$

for some $\lambda > 0$ (resp. for all $\lambda > 0$) if and only if

$$|H(f, \alpha)| \lesssim \sqrt{\alpha!} e^{-\frac{1}{r}\varphi^*(r|\alpha|)}$$

for some $r > 0$ (resp. for all $r > 0$) where

$$\varphi^*(x) = \sup_{y \geq 0} xy - \varphi(y).$$

Some examples

$\omega(x)$	$H(f,\alpha) \lesssim$	parameter
$\log_+(x)$	$1_{[0,1]}(r\lvert\alpha\rvert)$	
$\log(1+x)^{\frac{1}{1-2s}}$	$e^{-r\lvert\alpha\rvert^{\frac{1}{2s}}}$	$0 < s < 1/2$
$x^{\frac{2\sigma}{\sigma+1}}$	$r^{\lvert\alpha\rvert}(\alpha!)^{-\frac{1}{2\sigma}}$	$\sigma > 0$
x^2	$e^{-r\lvert\alpha\rvert}$	

Coordinate-wise time-frequency decay

Question: Can we still recover our quantitative result with only coordinate-wise time-frequency estimates?

In the limit case, the answer is positive:

Theorem (N., Toft, Vindas)

The following are equivalent for any $f \in L^2(\mathbb{R}^d)$:

- For every $j \in \{1, \dots, d\}$ and all $\lambda > 0$

$$|f(x)| \lesssim e^{-(\frac{1}{2}-\lambda)|x_j|^2}, \qquad |\widehat{f}(\xi)| \lesssim e^{-(\frac{1}{2}-\lambda)|\xi_j|^2}$$

- For all $\lambda > 0$

$$|f(x)| \lesssim e^{-(\frac{1}{2}-\lambda)|x|^2}, \qquad |\widehat{f}(\xi)| \lesssim e^{-(\frac{1}{2}-\lambda)|\xi|^2}$$

- For all $\varepsilon > 0$

$$|H(f, \alpha)| \lesssim \varepsilon^{|\alpha|}$$

Gaussian interpolation

A weight function ω satisfies Gaussian interpolation (of dimension d) if, for some $L, C > 0$,

$$\forall t \geq 0 \ \exists \sigma \in (1, d) : \quad t^{2\frac{d-\sigma}{d-1}} + \omega(t^\sigma) \leq L\omega(t) + C$$

For example:

- $\log(1+t)^{1+a}$, $a \geq 0$, **satisfies** Gaussian interpolation;
- t^a, $0 < a < 2$, **does not satisfy** Gaussian interpolation.

Lemma

A weight function ω satisfies Gaussian interpolation if and only if

$$e^{\frac{1}{r}\varphi^*(r|\alpha|)} \lesssim (\sqrt{\alpha!})^{1-\frac{1}{d}} \left(e^{\frac{1}{\mu}\varphi^*(\mu|\alpha|)}\right)^{\frac{1}{d}}$$

Results when Gaussian interpolation is satisfied

Theorem (N., Toft, Vindas)

Let ω be a weight function satisfying (α) and (δ) and Gaussian interpolation of dimension $d \geq 2$.

If $f \in L^2(\mathbb{R}^d)$ is such that for $\lambda > 0$ (resp. for all $\lambda > 0$)

$$|f(x)| \lesssim e^{-\frac{1}{2}|x_j|^2+\lambda\omega(|x_j|)}, \qquad |\widehat{f}(\xi)| \lesssim e^{-\frac{1}{2}|\xi_j|^2+\lambda\omega(|\xi_j|)}$$

then

$$|H(f,\alpha)| \lesssim \sqrt{\alpha!}e^{-\frac{1}{r}\varphi^*(r|\alpha|)}$$

for some $r > 0$ (resp. for all $r > 0$).

Counterexamples when Gaussian interpolation is not satisfied

Theorem (N., Toft, Vindas)

Let ω be a weight function satisfying (α) and (δ) but not Gaussian interpolation of dimension $d \geq 2$.

There exists a $f \in L^2(\mathbb{R}^d)$ such that for $\lambda > 0$ (resp. for all $\lambda > 0$)

$$|f(x)| \lesssim e^{-\frac{1}{2}|x_j|^2+\lambda\omega(|x_j|)}, \qquad |\widehat{f}(\xi)| \lesssim e^{-\frac{1}{2}|\xi_j|^2+\lambda\omega(|\xi_j|)}$$

but such that

$$\sup_{\alpha\in\mathbb{N}^d} \frac{|H(f,\alpha)|}{\sqrt{\alpha!}e^{-\frac{1}{\varepsilon}\varphi^*(\varepsilon|\alpha|)}} = \infty$$

for every $\varepsilon > 0$.

Thank you for your attention!

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