The parameter dependence of the Stieltjes moment problem

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The regularity of the Stieltjes moment problem

The Stieltjes moment problem

The Stieltjes moment problem asks whether one can find for any given sequence $(c_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ a non-decreasing function $\theta(t)$ on $[0,\infty)$ such that

$$\int_0^\infty t^n d\theta(t) = c_n, \qquad \forall n \in \mathbb{N}$$

- The problem was solved by Stieltjes in 1884.
- In 1938 Boas and Polya found that there always exists a function θ of *bounded variation*.
- In 1989 Durán showed that one may always choose $d\theta(t) = f(t)dt$ with $f \in \mathcal{S}(\mathbb{R}_+)$, with

$$\mathcal{S}(\mathbb{R}_+) = \{g \in \mathcal{S}(\mathbb{R}) \mid \operatorname{supp} g \subseteq [0,\infty)\}.$$

- In 1991 he improved the result by giving a characterization of the solvability of the Stieltjes moment problem for general exponents $\{z_n\}_{n\in\mathbb{N}}$.

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Question: How regular are the solutions of the Stieltjes moment problem?

Observation: One cannot let the solutions depend both linearly and continuously on $(c_n)_{n\in\mathbb{N}}$.

Alternative: Let Λ be an index set and $\omega_j : \Lambda \to \mathbb{R}_+$ such that $\omega_{j+1}(\lambda) \leq \omega_j(\lambda)$.

If we have $(c_{n,\lambda}) \in \mathbb{C}^{\mathbb{N}\times\Lambda}$ satisfying

$$\forall n \in \mathbb{N}\ \exists j \in \mathbb{N}\ :\ \sup_{\lambda\in\Lambda} |c_{n,\lambda}| w_j(\lambda) < \infty$$

do there exist corresponding solutions $(f_\lambda)_\lambda$ with global regularity w.r.t. the ω_j's?

We studied this question in our recent preprint:

A. Debrouwere, L. Neyt, *The vector-valued Stieltjes moment problem with general exponents*, preprint. Available online at arXiv:2401.00497.

Its characterization is given in the ensuing slide. The goal of this presentation is to discuss how we obtained this result.

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Theorem (Debrouwere, N.)

The following are equivalent:

(i) $\exists j\ \forall k \geq j\ \exists l \geq k\ \forall \lambda \in \Lambda\ :\ w_j(\lambda)w_l(\lambda) \lesssim w_k(\lambda)^2$.

(ii) For every sequence $(c_{n,\lambda}) \in \mathbb{C}^{\mathbb{N}\times\Lambda}$ satisfying

$$\forall n \in \mathbb{N}\ \exists j \in \mathbb{N}\ :\ \sup_{\lambda\in\Lambda} |c_{n,\lambda}| w_j(\lambda) < \infty \tag{1}$$

there exist measurable functions $f_\lambda : \mathbb{R}_+ \to \mathbb{C}$, $\lambda \in \Lambda$, such that

$$\forall n \in \mathbb{N}\ \exists j \in \mathbb{N}\ :\ \sup_{\lambda\in\Lambda} \int_0^\infty |t^n f_\lambda(t)|dt\ w_j(\lambda) < \infty$$

and

$$\int_0^\infty t^n f_\lambda(t)dt = c_{n,\lambda}, \qquad \forall n \in \mathbb{N}, \lambda \in \Lambda. \tag{2}$$

(iii) We may take the solutions $f_\lambda \in \mathcal{S}(\mathbb{R}_+)$ such that

$$\forall k \in \mathbb{N}\ \exists j \in \mathbb{N}\ :\ \sup_{\lambda\in\Lambda} \max_{0\leq l\leq k} \sup_{t\in\mathbb{R}_+} |t^{k+l+1} f_\lambda^{(l)}(t)|\omega_j(\lambda)$$

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Example (Examples of weights)

Suppose $(\Lambda, \|\cdot\|)$ is a normed space and $\omega : [0,\infty) \to [0,\infty)$. E.g.,

* $\omega = \log(1 + |\cdot|)$;
* $\omega = |\cdot|^s$ for $s > 0$.

The weights $w_j(\lambda) = e^{-j\omega(\|\lambda\|)}$ always do satisfy the condition (i) of the previous theorem.

The weights $w_j(\lambda) = e^{\frac{1}{j+1}\omega(\|\lambda\|)}$ do not satisfy the condition (i) of the previous theorem.

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The problem of parameter dependence

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Suppose we have a solvable problem

$$P(x) = y \in H, \qquad x \in G. \tag{3}$$

Given data $(y_\lambda)_\lambda \subset H$ depending *regularly* on the parameter λ, do there exist corresponding solutions $x_\lambda \in G$ such that

$$P(x_\lambda) = y_\lambda, \qquad \forall \lambda,$$

depending on λ with the same regularity?

Example (Examples of regularity)

- Smoothly: $\Omega \ni \lambda \to y_\lambda \in H$ is a smooth function ($\Omega \subseteq \mathbb{R}^d$ open)
- Holomorphically: $\Omega \ni \lambda \to y_\lambda \in H$ is a holomorphic function ($\Omega \subseteq \mathbb{C}^d$ open)
- Weighted discrete way: $\Lambda \ni \lambda \to y_\lambda \in H$ is a weighted bounded sequence

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The problem of parameter dependence – reformulation

The problem thus becomes solving a "*vector-valued version of P*".

For two locally convex Hausdorff spaces (=lcHs) E and F we put

$$E\varepsilon F = \{ \text{ continuous linear maps } E'_c \to F \ \}$$

Note that $E\varepsilon F \cong F\varepsilon E$ by transposition.

The problem of parameter dependence

For which lcHs E is the map

$$E\varepsilon G \to E\varepsilon H, \quad L \mapsto P \circ L$$

surjective?

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We will consider this problem when G, H are Fréchet spaces.

When E and F are complete and one of them is nuclear we have that:

$$E\varepsilon F \cong E\widehat{\otimes}_\pi F$$

- Fact: the π-tensor product of two surjective continuous linear maps between Fréchet spaces is again surjective.
  $\hookrightarrow$ The problem becomes trivial when G, H are nuclear and E is a Fréchet space.
- Assumption: we will now always assume E
  * is locally complete
  * has a fundamental sequence of bounded sets $\{B_N\}$

For any Schwartz Fréchet space F we have that:

$$E\varepsilon F \cong F\varepsilon E \cong L(F_b', E)$$

and F_b' is an (LB)-space.

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Weak E-acyclicity

In

A. Debrouwere, L. Neyt, *An extension result for (LB)-spaces and the surjectivity of tensorized mappings*, Annali di Matematica (2024). DOI:10.1007/s10231-023-01420-0

we characterized those (LB)-spaces Z that are weakly E-acyclic, i.e., when for each short exact sequence

$$0 \longrightarrow X \xrightarrow{\iota} Y \longrightarrow Z \longrightarrow 0$$

of (LB)-spaces it holds that the map

$$\iota_* : L(Y, E) \mapsto L(X, E), \quad T \mapsto T \circ \iota$$

is surjective.

The characterization of weak E-acyclicity

Theorem (Debrouwere, N., 2024)

For a nuclear (LB)-space $Z = \varinjlim_n Z_n$, the following are equivalent:

- (E, Z) satisfies (S), i.e.,

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Let G, H be Fréchet-Schwartz spaces. Suppose $P : G \to H$ is a surjective continuous linear map. If

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Vector-valued Eidelheit families

Let G be a Fréchet space and $(x'_n)_{n\in\mathbb{N}} \subseteq G'$. Consider the problem:

$$P : G \to \mathbb{C}^{\mathbb{N}}, \quad x \mapsto (\langle x'_n, x\rangle)_{n\in\mathbb{N}}$$

Eidelheit (1936) characterized when P is solvable and in this case $(x'_n)_{n\in\mathbb{N}}$ is called an Eidelheit sequence on G.

Example (The Stieltjes moment problem)

Note that the mappings

$$\mathcal{M}_n : \mathcal{S}(\mathbb{R}_+) \to \mathbb{C}, \quad f \mapsto \int_0^\infty t^n f(t)dt$$

form and Eidelheit sequence on $\mathcal{S}(\mathbb{R}_+)$.

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A Fréchet space F with fundamental increasing sequence of seminorms $(\|\cdot\|_n)_{n\in\mathbb{N}}$ is said to satisfy $(\underline{DN})$ if

$$\exists n \in \mathbb{N}\ \forall m \geq n\ \exists k \geq m, \theta \in (0,1), C > 0\ \forall x \in F :$$
$$\|x\|_m \leq C\|x\|_n^{\theta}\|x\|_k^{1-\theta}$$

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Let G be a nuclear Fréchet space satisfying $(\underline{DN})$. Let $(x_n') \subset G'$ be an Eidelheit sequence such that

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satisfies (Ω). The following are equivalent.

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A Fréchet algebra $(F, *)$ is called *m*-convex if there exists an increasing fundamental sequence of seminorms $(\|\cdot\|_n)_{n\in\mathbb{N}}$ on F such that

$$\|x * y\|_n \leq \|x\|_n \|y\|_n, \qquad \forall x, y \in F,\ n \in \mathbb{N}.$$

Theorem (Braun, 1987)

Let $(G, *)$ be an *m*-convex Fréchet algebra and $(x_n') \subset G'$ be an Eidelheit sequence such that each $x_n' : (G, *) \to \mathbb{C}$ is an algebra homomorphism. Then, if G satisfies (Ω), so does

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The parameter dependence of the Stieltjes moment problem

By $\mathcal{S}(\mathbb{R}_+; E)$ we mean the space of all $\mathbf{f} \in C^\infty(\mathbb{R}_+; E)$ such that for all $k \in \mathbb{N}$ and continuous seminorm p on E:

$$\max_{0 \le l \le k} \sup_{t>0} t^{k+l+1} p(\mathbf{f}^{(l)}(t)) < \infty.$$

Theorem (Debrouwere, N.)

The following are equivalent:

(i) E satisfies (A).

(ii) for every $(e_n)_{n\in\mathbb{N}} \subseteq E^{\mathbb{N}}$ there exists a locally Pettis integrable function $\mathbf{f} : \mathbb{R}_+ \to E$ such that

$$\int_0^\infty t^n \mathbf{f}(t) dt = e_n, \qquad \forall n \in \mathbb{N}. \tag{4}$$

(iii) for every $(e_n)_{n\in\mathbb{N}} \subseteq E^{\mathbb{N}}$ there exists a $\mathbf{f} \in \mathcal{S}(\mathbb{R}_+; E)$ such that (4) holds.

The result of the introduction is then obtained by taking

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Using the previous results, it suffices to verify the following facts:

- $\mathcal{S}(\mathbb{R}_+) \cong s$ using Hermite expansions. In particular, $\mathcal{S}(\mathbb{R}_+)$ is nuclear and satisfies $(\underline{DN})$ and (Ω).
- Consider the Mellin convolution

$$f *_M g(t) = \int_0^\infty f(x) g\left(\frac{t}{x}\right) \frac{dx}{x}.$$

  Then, $(\mathcal{S}(\mathbb{R}_+), *_M)$ is an m-convex Fréchet algebra and

$$\mathcal{M}_n(f *_M g) = \mathcal{M}_n(f) \cdot \mathcal{M}_n(g).$$

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Thank you for your attention!

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