Quantitative uncertainty principles for time-frequency Gaussian decay

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Based on collaborative work with Joachim Toft and Jasson Vindas

Workshop on Quantum Harmonic Analysis 2026
Växjö, Sweden – March 24, 2026

FWF Österreichischer Wissenschaftsfonds

Hardy's uncertainty principle

Wiener observed that a function $f \in L^2(\mathbb{R}^d)$ and its Fourier transform

$$\mathcal{F}\{f\}(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^d} f(t) e^{-i\langle t,\xi\rangle}\, dt$$

cannot both be too well-localized.

Theorem (Hardy, 1933)

Let $f \in L^2(\mathbb{R})$ be such that for some $\eta > 0$

$$|f(x)| \lesssim e^{-\frac{\eta}{2}x^2} \quad \text{and} \quad |\widehat{f}(\xi)| \lesssim e^{-\frac{\eta}{2}\xi^2}.$$

Then:

- $\eta > 1 \Longrightarrow f \equiv 0$;
- $\eta = 1 \Longrightarrow f(x) = ce^{-\frac{1}{2}|x|^2}$ for some $c \in \mathbb{C}$;
- $\eta < 1 \Longrightarrow$ The space of all functions f satisfying the time-frequency decay is dense in $L^2(\mathbb{R})$.

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- $\eta < 1 \implies$ The space of all functions f satisfying the time-frequency decay is dense in $L^2(\mathbb{R})$.

Hardy's uncertainty principle – Quantified

Hardy quantified further:

Theorem (Hardy, 1933)

For any $f \in L^2(\mathbb{R})$: The time-frequency decay

$$|f(x)| \lesssim (1+|x|)^N e^{-\frac{1}{2}x^2} \quad \text{and} \quad |\widehat{f}(\xi)| \lesssim (1+|\xi|)^N e^{-\frac{1}{2}\xi^2}$$

holds for some $N \in \mathbb{N}$ if and only if

$$f(x) = P(x)e^{-\frac{1}{2}x^2}$$

for some polynomial P of degree at most N.

Hardy's uncertainty principle – Multi-dimensional version

Multi-dimensional versions of Hardy's uncertainty principle have been established. In particular, it was observed that coordinate-wise estimates suffice.

Theorem (Bonami, Demange, Jaming, 2003)

For any $f \in L^2(\mathbb{R}^d)$: For every $j \in \{1, \dots, d\}$, the time-frequency decay

$$|f(x)| \lesssim (1 + |x_j|)^N e^{-\frac{1}{2}|x_j|^2} \quad \text{and} \quad |\widehat{f}(\xi)| \lesssim (1 + |\xi_j|)^N e^{-\frac{1}{2}|\xi_j|^2}$$

holds for some $N \in \mathbb{N}$ if and only if

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for some polynomial P of degree at most N in each variable.

Question: Which structure does time-frequency of the form

$$|f(x)|, |\widehat{f}(\xi)| \lesssim e^{\lambda \omega(|\cdot|) - \frac{1}{2}|\cdot|^2}, \qquad \lambda > 0,$$

impose on the function f when ω is a weight function dominated by t^2?

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Hermite coefficients

Consider the Hermite functions

$$h_\alpha(x) = (-1)^{|\alpha|}\pi^{-\frac{d}{4}}(2^{|\alpha|}\alpha!)^{-\frac{1}{2}}e^{\frac{1}{2}|x|^2}\frac{\partial^\alpha}{\partial x^\alpha}[e^{-|x|^2}], \qquad \alpha \in \mathbb{N}^d.$$

They form an orthonormal basis in $L^2(\mathbb{R}^d)$. So, for any $f \in L^2(\mathbb{R}^d)$,

$$f = \sum_{\alpha\in\mathbb{N}^d} H(f,\alpha)h_\alpha, \qquad \text{with} \quad H(f,\alpha) = (f, h_\alpha)_{L^2}.$$

We find the unitary map

$$L^2(\mathbb{R}^d) \to \ell^2(\mathbb{N}^d), \quad f \mapsto (H(f,\alpha))_{\alpha\in\mathbb{N}^d}.$$

Note that

$$|H(\widehat{f},\alpha)| = |H(f,\alpha)|, \qquad f \in L^2(\mathbb{R}^d).$$

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Time-frequency decay vs. Hermite series expansions

Note that for a polynomial P of degree N, we have

$$f(x) = P(x)e^{-\frac{1}{2}|x|^2} \quad \Longleftrightarrow \quad H(f,\alpha) = 0 \text{ when } |\alpha| > N.$$

The Hermite functions also detect less sharp time-frequency decay:

- Schwartz (1950) showed:

$$\forall \eta > 0: \quad |f(x)|, |\widehat{f}(\xi)| \lesssim (1+|\cdot|)^{-\eta} \quad \Longleftrightarrow \quad \forall r > 0: \quad |H(f,\alpha)| \lesssim |\alpha|^{-r}$$

- De Bruijn (1973) and Zhang (1963) independently showed:

$$\exists \eta \leq 1: \quad |f(x)|, |\widehat{f}(\xi)| \lesssim e^{-\frac{\eta}{2}|\cdot|^2} \quad \Longleftrightarrow \quad \exists r < 1: \quad |H(f,\alpha)| \lesssim r^{-|\alpha|}$$

See also the work of Pilipović and Langenbruch.

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Aim of the talk

In this talk, we will discuss how time-frequency of the form

$$|f(x)|, |\widehat{f}(\xi)| \lesssim e^{\lambda\omega(|\cdot|)-\frac{1}{2}|\cdot|^2}, \qquad \lambda > 0,$$

imposes specific bounds on the Hermite coefficients of f.

- We characterized this behavior in the one-dimensional case in

  L. Neyt, J. Toft, J. Vindas, *Hermite expansions for spaces of functions with nearly optimal time-frequency decay*, J. Funct. Anal. **288** (2025), 110706.

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From time-frequency decay to bounds on entire functions and back

Weight functions

In this talk, a weight function ω stands for:

- an unbounded non-decreasing function $[0,\infty) \to [0,\infty)$ going to ∞
- satisfying
$$\omega(2t) = O(\omega(t))$$
- such that $\varphi(t) = \omega(e^t)$ is convex.

We will consider the *convex conjugate* φ^* of φ

$$\varphi^*(x) = \sup_{y \geq 0} xy - \varphi(y).$$

Then, $\frac{1}{r}\varphi^*(r\cdot)$ is non-decreasing in r.

Either $\omega \asymp \log_+$ or $\log x = o(\omega(x))$.

Examples: $\omega(t) = t^a \log_+^b t$ for $a > 0$ and $b \geq 0$ or $a = 0$ and $b \geq 1$.

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Associated indices

To effectively estimate our bounds, we introduce the following indices in function of the weight function ω:

$$\mathfrak{m}(\omega) = \inf\{L \in [1,\infty) \mid \exists C > 0 \ \forall t, s \geq 0 : \omega(t+s) \leq L[\omega(t) + \omega(s)] + C\}$$

and, for $\kappa > 0$,

$$\mathfrak{m}_\kappa(\omega) = \inf\{L \in (0,\infty) \mid \exists C > 0 \ \forall t \geq 0 : \omega(\kappa t) \leq L\omega(t) + C\}$$

Some properties:

- $\kappa \mapsto \mathfrak{m}_\kappa(\omega)$ is submultiplicative $\longrightarrow \mathfrak{m}_\kappa(\omega) \leq \mathfrak{m}_2(\omega)^{\lceil \log_2 \kappa \rceil}$ for $\kappa \geq 1$
- $\mathfrak{m}(\omega) \leq \mathfrak{m}_2(\omega) \leq 2\mathfrak{m}(\omega)$
- if $\mathfrak{m}_{\kappa_0}(\omega) = 1$ for some $\kappa_0 > 1$, then $\mathfrak{m}(\omega) = \mathfrak{m}_\kappa(\omega) = 1$ for all $\kappa > 0$
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The Bargmann transform

The Bargmann transform gives a unitary map

$$\mathcal{B}_d : L^2(\mathbb{R}^d) \to \mathscr{F}_2, \quad f \mapsto \left(z \mapsto \mathcal{B}_d f(z) = \sum_{\alpha \in \mathbb{N}^d} \frac{H(f,\alpha)}{\sqrt{\alpha!}} z^\alpha \right)$$

with

$$\mathscr{F}_2 = \{F \in \mathcal{A}(\mathbb{C}^d) \mid \int_{\mathbb{C}^d} |F(z)|^2 e^{-|z|^2} dz < \infty\}$$

the Fock space.

Note that

$$H(f,\alpha) = \frac{1}{\sqrt{\alpha!}} [\mathcal{B}_d f]^{(\alpha)}(0), \qquad \alpha \in \mathbb{N}^d$$

Explicitly, the integral transform is given by

$$\mathcal{B}_d f(z) = \pi^{-\frac{d}{4}} \int_{\mathbb{R}^d} f(t) e^{-\frac{1}{2}\left(\langle z,z\rangle + |t|^2\right) + \sqrt{2}\langle z,t\rangle} dt.$$

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Hermite characterization of weighted entire functions

Proposition

For any entire function F, the bound

$$|F(z)| \lesssim e^{\lambda\omega(|z|)}$$

holds for some $\lambda > 0$ if and only if there exists a $f \in L^2(\mathbb{R}^d)$ such that

$$|H(f,\alpha)| \lesssim \sqrt{\alpha!}e^{-\frac{1}{r}\varphi^*(r|\alpha|)}$$

for some $r > 0$ and

$$F = \mathfrak{B}_d f$$

The Bargmann transform and time-frequency decay

Lemma

Let ω be a weight function such that $\omega(t) = o(t^2)$ (resp. $\omega(t) = t^2$).

For a function $f \in L^2(\mathbb{R}^d)$, consider the statements:

(i) $|\mathcal{B}_d f(z)| \lesssim e^{\lambda\omega(|z|)}$ for some $\lambda > 0$ (resp. for $\lambda < 1/2$).

(ii) For some $\lambda > 0$ (resp. for $\lambda < 1/2$)

$$|f(x)| \lesssim e^{-\frac{1}{2}|x|^2+\lambda\omega(|t|)} \quad \text{and} \quad |\widehat{f}(\xi)| \lesssim e^{-\frac{1}{2}|\xi|^2+\lambda\omega(|\xi|)}.$$

(iii) For some $\lambda > 0$ (resp. for $\lambda < 1/2$), for $z = x + i\xi$,

$$|\mathcal{B}_d f(z)| \lesssim e^{\frac{1}{2}|x|^2+\lambda\omega(|\xi|)} \quad \text{and} \quad |\mathcal{B}_d f(z)| \lesssim e^{\frac{1}{2}|\xi|^2+\lambda\omega(|x|)}$$

$\hookrightarrow$ In particular, $|\mathcal{B}_d f(\cdot)| \lesssim e^{\lambda\omega(|\cdot|)}$ on $\mathbb{R}^d \cup i\,\mathbb{R}^d$.

Then, (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii).

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$$|f(x)| \lesssim e^{-\frac{1}{2}|x|^2+\lambda\omega(|t|)} \quad \text{and} \quad |\widehat{f}(\xi)| \lesssim e^{-\frac{1}{2}|\xi|^2+\lambda\omega(|\xi|)}.$$

(iii) For some $\lambda > 0$ (resp. for $\lambda < 1/2$), for $z = x + i\xi$,

$$|\mathfrak{B}_d f(z)| \lesssim e^{\frac{1}{2}|x|^2+\lambda\omega(|\xi|)} \quad \text{and} \quad |\mathfrak{B}_d f(z)| \lesssim e^{\frac{1}{2}|\xi|^2+\lambda\omega(|x|)}$$

$\hookrightarrow$ In particular, $|\mathcal{B}_d f(\cdot)| \lesssim e^{\lambda\omega(|\cdot|)}$ on $\mathbb{R}^d \cup i\,\mathbb{R}^d$.

Then, (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii).

The Bargmann transform and time-frequency decay

Lemma

Let ω be a weight function such that $\omega(t) = o(t^2)$ (resp. $\omega(t) = t^2$).

For a function $f \in L^2(\mathbb{R}^d)$, consider the statements:

(i) $|\mathfrak{B}_d f(z)| \lesssim e^{\lambda\omega(|z|)}$ for some $\lambda > 0$ (resp. for $\lambda < 1/2$).

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$$|f(x)| \lesssim e^{-\frac{1}{2}|x|^2+\lambda\omega(|t|)} \quad \text{and} \quad |\widehat{f}(\xi)| \lesssim e^{-\frac{1}{2}|\xi|^2+\lambda\omega(|\xi|)}.$$

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Then, (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii).

A (multi-dimensional) Phragmén-Lindelöf principle

A classical Phragmén-Lindelöf type result

We write

$$\mathcal{Q}_1 = \{z = x + i\xi \in \mathbb{C} \mid x, \xi > 0\}.$$

Note: We are working in dimension one.

Lemma (Classical Phragmén-Lindelöf principle on quadrants)

Suppose F is analytic in $\mathcal{Q}_1$ with continuous extension to $\overline{\mathcal{Q}_1}$ such that

$$|F(v)| \leq M, \qquad v \in [0, \infty) \cup i[0, \infty)$$

and, for all $\varepsilon > 0$,

$$|F(z)| \lesssim_\varepsilon e^{\varepsilon |z|^2}, \qquad z \in \mathcal{Q}_1$$

Then

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An optimal result

Theorem (N., Toft, Vindas, 2025)

The following statements are equivalent:

(I) $\int_x^\infty \frac{\omega(t)}{t^3}\,dt = O\left(\frac{\omega(x)}{x^2}\right)$.

(II) There exists $A > 0$ such that for any $\lambda > 0$:
There exists $B > 0$ such that for any F analytic in $\mathcal{Q}_1$ with continuous extension to $\overline{\mathcal{Q}_1}$ for which

$$|F(v)| \leq Me^{\lambda\omega(|v|)}, \qquad v \in [0,\infty) \cup i[0,\infty)$$

and, for all $\varepsilon > 0$,

$$|F(z)| \lesssim_\varepsilon e^{\varepsilon|z|^2}, \qquad z \in \overline{\mathcal{Q}_1}$$

then

$$|F(z)| \leq BMe^{A\lambda\omega(|z|)}, \qquad z \in \overline{\mathcal{Q}_1}$$

$$\mathrm{PL}_{\frac{\pi}{2}}(\omega) = \inf\{A \in [1,\infty) \mid \text{(II) holds for } A\}$$

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Sketch of the proof

Using the Poisson transform

$$P_\omega(x,y) = \frac{x}{\pi}\int_{-\infty}^{\infty} \frac{\omega(\sqrt{t})}{x^2+(y-t)^2}\,dt,$$

one constructs F_ω that is analytic on $\mathcal{Q}_1$ with continuous extension to $\overline{\mathcal{Q}_1}$ satisfying the following properties:

(i) For some $c, \nu_1 > 0$

$$\forall \varepsilon > 0: \quad ce^{\nu_1 \omega(|z|)} \leq |F_\omega(z)| \lesssim e^{\varepsilon |z|^2}, \qquad z \in \mathcal{Q}_1.$$

(ii) $|F_\omega(v)| \lesssim e^{\mu\omega(|z|)}$ for $v \in [0,\infty) \cap i[0,\infty)$ for some $\mu > 0$.

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(I) $\Rightarrow$ **(II)** : Apply the classical Phragmén-Lindelöf result to $G = F \cdot F_\omega^{-\lambda/\nu_1}$.

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Towards the multidimensional case

Proposition (N., Toft, Vindas)

Let F be an entire function on $\mathbb{C}^d$, if

$$\limsup_{s\in\mathbb{C},\ |s|\to\infty} \frac{\log|F(su)|}{\omega(|s|)} < \infty, \qquad \forall u \in \mathbb{S}^{d-1}$$

then

$$\limsup_{z\in\mathbb{C}^d,\ |z|\to\infty} \frac{\log|F(z)|}{\omega(|z|)} < \infty$$

Proof of the Proposition

For any $u \in \mathbb{S}^{d-1}$, define the function

$$F_u(s) = F(us) = \sum_{\alpha \in \mathbb{N}^d} \frac{F^{(\alpha)}(0)}{\alpha!} \prod_{j=1}^{d} (u_j s)^{\alpha_j} = \sum_{n=0}^{\infty} \left(\sum_{|\alpha|=n} \frac{F^{(\alpha)}(0)}{\alpha!} u^{\alpha} \right) s^n$$

Using the Cauchy inequalities, we have for some $\lambda > 0$

$$\left| \sum_{|\alpha|=n} \frac{F^{(\alpha)}(0)}{\alpha!} u^{\alpha} \right| \lesssim e^{-\lambda \varphi^*\left(\frac{|\alpha|}{\lambda}\right)}$$

Therefore, the homogenous polynomial

$$P_n(X) = e^{\lambda \varphi^*\left(\frac{|\alpha|}{\lambda}\right)} \sum_{|\alpha|=n} \frac{F^{(\alpha)}(0)}{\alpha!} X_1^{\alpha_1} \cdots X_d^{\alpha_d}$$

is bounded on $\mathbb{S}^{d-1}$. By a result due to Kellogg (1928), we then find

$$\frac{|F^{(\alpha)}(0)|}{\alpha!} \lesssim d^{|\alpha|} e^{\lambda \varphi^*\left(-\frac{|\alpha|}{\lambda}\right)}$$

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A multi-dimensional Phragmén-Lindelöf principle

Theorem (N., Toft, Vindas)

Let ω satisfy

$$\int_x^\infty \frac{\omega(t)}{t^3}\,dt = O\left(\frac{\omega(x)}{x^2}\right)$$

If F is an entire function on $\mathbb{C}^d$ such that, for some $\lambda > 0$,

$$|F(v)| \lesssim e^{\lambda\omega(|v|)}, \qquad v \in \mathbb{R}^d \cup i\mathbb{R}^d$$

and, for every $\varepsilon > 0$,

$$|F(z)| \lesssim e^{\varepsilon|z|^2}, \qquad z \in \mathbb{C}^d$$

then, for some $\mu > 0$,

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Estimates for uniform time-frequency decay

Maximal case

Theorem (N., Toft, Vindas, $d = 1$ – 2025; $d \geq 2$ preprint)

If $f \in L^2(\mathbb{R}^d)$ satisfies

$$|f(x)| \lesssim e^{-(\frac{1}{2}-\lambda)|x|^2} \quad \text{and} \quad |\widehat{f}(\xi)| \lesssim e^{-(\frac{1}{2}-\lambda)|\xi|^2}$$

for some $0 < \lambda < 1/2$, then

$$|H(f,\alpha)| \lesssim r^{-|\alpha|}, \qquad r < \frac{1}{d}(2\lambda)^{-\frac{1}{4}}$$

Weighted case

Theorem (N., Toft, Vindas, $d = 1$ – 2025; $d \geq 2$ preprint)

Suppose ω satisfies

$$\int_x^\infty \frac{\omega(y)}{y^3}\,dy = O\left(\frac{\omega(x)}{x^2}\right)$$

For the constant

$$\mathfrak{C}_{1,d}(\omega) = \mathfrak{m}(\omega)\mathfrak{m}_{\frac{d}{\sqrt{2}}}(\omega)\mathrm{PL}_{\frac{\pi}{2}}(\omega)$$

if a function $f \in L^2(\mathbb{R}^d)$ satisfies

$$|f(x)| \lesssim e^{-\frac{1}{2}|x|^2+\lambda\omega(|x|)} \quad \text{and} \quad |\widehat{f}(\xi)| \lesssim e^{-\frac{1}{2}|\xi|^2+\lambda\omega(|\xi|)}$$

for some $\lambda > 0$, then

$$|H(f,\alpha)| \lesssim \sqrt{\alpha!}e^{-\frac{1}{r}\varphi^*(r|\alpha|)}, \qquad 0 < r < (\mathfrak{C}_{1,d}(\omega)\lambda)^{-1}$$

Some examples

$\omega(x)$	$\sqrt{\alpha!}e^{-\frac{1}{r}\varphi^*(r\lvert\alpha\rvert)} \asymp$	$\mathfrak{C}_{1,d}(\omega)$
$\log_+(x)$	$1_{[0,1]}(r\lvert\alpha\rvert)$	1
$\log_+(x)^{\frac{s}{s-1}}$, $s>1$	$e^{-r\lvert\alpha\rvert^s}$	1
$x^{\frac{2\sigma}{\sigma+1}}$, $0<\sigma\leq 1$	$r^{\lvert\alpha\rvert}(\alpha!)^{-\frac{1}{2\sigma}}$	$d^{\frac{2\sigma}{\sigma+1}}2^{-\frac{\sigma}{\sigma+1}}\sec\left(\frac{\sigma}{\sigma+1}\frac{\pi}{2}\right)$
$x^{\frac{2\sigma}{\sigma+1}}$, $\sigma>1$	$r^{\lvert\alpha\rvert}(\alpha!)^{-\frac{1}{2\sigma}}$	$d^{\frac{2\sigma}{\sigma+1}}2^{-\frac{1}{\sigma+1}}\sec\left(\frac{\sigma}{\sigma+1}\frac{\pi}{2}\right)$

Estimates for coordinate-wise time-frequency decay

First estimates

If only coordinate-wise estimates are known, by iteration, we find the following estimates:

Lemma

Suppose ω satisfies

$$\int_x^\infty \frac{\omega(y)}{y^3}dy = O\left(\frac{\omega(x)}{x^2}\right)$$

If $f \in L^2(\mathbb{R}^d)$ satisfies, for some $\lambda > 0$,

$$|f(x)| \lesssim e^{-\frac{1}{2}|x_j|^2+\lambda\omega(|x_j|)} \quad \text{and} \quad |\widehat{f}(\xi)| \lesssim e^{-\frac{1}{2}|\xi_j|^2+\lambda\omega(|\xi_j|)}$$

for all $j \in \{1, \dots, d\}$, then

$$|H(f,\alpha)| \lesssim (\alpha!)^{\frac{1}{2d}} e^{-\frac{1}{r}\varphi^*\left(\frac{r|\alpha|}{d}\right)}$$

for some $r > 0$.

Quadratic interpolation

We say a weight function ω satisfies quadratic interpolation (of dimension d) if, for some $L, C > 0$,

$$\forall t \geq 0 \ \exists \sigma_t \in [1, d] : \quad t^{2\frac{d-\sigma_t}{d-1}} + \omega(t^{\sigma_t}) \leq L\omega(t) + C$$

For example:

- $\log(1+t)^{1+a}$, $a \geq 0$, **satisfies** Gaussian interpolation;
- t^a, $0 < a < 2$, **does not satisfy** Gaussian interpolation;
- t^2 **satisfies** Gaussian interpolation.

Lemma

A weight function ω satisfies quadratic interpolation if and only if

$$e^{\frac{1}{r}\varphi^*(r|\alpha|)} \lesssim (\sqrt{\alpha!})^{1-\frac{1}{d}} e^{\frac{1}{\mu}\varphi^*\left(\frac{\mu|\alpha|}{d}\right)}$$

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Results for quadratic interpolation – $\omega(t) = t^2$

Theorem (N., Toft, Vindas)

If a function $f \in L^2(\mathbb{R}^d)$ satisfies

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for some $0 < \lambda < 1/2$ and all $j \in \{1, \ldots, d\}$, then

$$|H(f, \alpha)| \lesssim r^{-|\alpha|}, \qquad r < (2\lambda)^{-\frac{1}{4d}}$$

Results for quadratic interpolation – General

Theorem (N., Toft, Vindas)

Let ω be a weight function satisfying quadratic interpolation of dimension $d \geq 2$ and

$$\int_x^\infty \frac{\omega(y)}{y^3}\,dy = O\left(\frac{\omega(x)}{x^2}\right)$$

The exists a function $\lambda \mapsto \mathfrak{C}_{2,d}(\omega,\lambda)$ such that if a function $f \in L^2(\mathbb{R}^d)$ satisfies

$$|f(x)| \lesssim e^{-\frac{1}{2}|x_j|^2+\lambda\omega(|x_j|)} \quad \text{and} \quad |\widehat{f}(\xi)| \lesssim e^{-\frac{1}{2}|\xi_j|^2+\lambda\omega(|\xi_j|)}$$

for some $\lambda > 0$ and all $j \in \{1,\dots,d\}$, then

$$|H(f,\alpha)| \lesssim \sqrt{\alpha!}e^{-\frac{1}{r}\varphi^*(r|\alpha|)}, \qquad 0 < r < (\mathfrak{C}_{2,d}(\omega,\lambda))^{-1}$$

Results for quadratic interpolation – $\omega(t) = \log_+^{\frac{s}{s-1}} t$

Our result recovers the coordinate-wise quantitative result of Bonami, Demanghe, and Jaming. But also extends it:

Theorem (N., Toft, Vindas)

Let $s > 1$. If a function $f \in L^2(\mathbb{R}^d)$ satisfies

$$|f(x)| \lesssim e^{-\frac{1}{2}|x_j|^2 + \lambda \log_+^{\frac{s}{s-1}}(|x_j|)} \quad \text{and} \quad |\widehat{f}(\xi)| \lesssim e^{-\frac{1}{2}|\xi_j|^2 + \lambda \log_+^{\frac{s}{s-1}}(|\xi_j|)}$$

for some $\lambda > 0$ and all $j \in \{1, \dots, d\}$, then

$$|H(f, \alpha)| \lesssim e^{-(r|\alpha|)^s}, \qquad 0 < r < \frac{1}{d} \frac{s-1}{s} \left(\frac{1}{s-1}\right)^{\frac{1}{s}} \lambda^{-\frac{s-1}{s}}$$

Counterexamples – General

Theorem (N., Toft, Vindas)

Let ω be a weight function that does not satisfy quadratic interpolation of dimension $d \geq 2$.

There exists a $f \in L^2(\mathbb{R}^d)$ such that, for all $\lambda > 0$,

$$|f(x)| \lesssim e^{-\frac{1}{2}|x_j|^2+\lambda\omega(|x_j|)}, \qquad |\widehat{f}(\xi)| \lesssim e^{-\frac{1}{2}|\xi_j|^2+\lambda\omega(|\xi_j|)}$$

but such that

$$\sup_{\alpha\in\mathbb{N}^d} \frac{|H(f,\alpha)|}{\sqrt{\alpha!}e^{-\frac{1}{\varepsilon}\varphi^*(\varepsilon|\alpha|)}} = \infty$$

for every $\varepsilon > 0$.

Proof

Since ω does not satisfy quadratic interpolation, one may construct a sequence (r_n) of positive numbers going to ∞ such that

$$\sup_{n\in\mathbb{N}} \frac{((dn)!)^{\frac{1}{2d}} e^{-\frac{1}{r_n}\varphi^*(r_n n)}}{\sqrt{(dn)!}e^{-\frac{1}{\varepsilon}\varphi^*(\varepsilon n)}} = \infty.$$

We then construct the entire function

$$F(z) = \sum_{n=0}^{\infty} 2^{-nd} \frac{z_1^n \cdots z_d^n}{(n!)^{\frac{d-1}{2}} e^{\frac{1}{r_n}\ \varphi^*(r_n n)}}$$

There exists a $f \in L^2(\mathbb{R}^d)$ such that

$$F = \mathcal{B}_d f$$

so that f satisfies all the conditions.

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Counterexamples – $\omega(t) = t^{\frac{2\sigma}{\sigma+1}}$

Theorem (N., Toft, Vindas)

Let $\sigma > 0$. There exists a $f \in L^2(\mathbb{R}^d)$ such that, for all $\lambda > 0$,

$$|f(x)| \lesssim e^{-\frac{1}{2}|x_j|^2+\lambda|x_j|^{\frac{2\sigma}{\sigma-1}}} \quad \text{and} \quad |\widehat{f}(\xi)| \lesssim e^{-\frac{1}{2}|\xi_j|^2+\lambda|\xi_j|^{\frac{2\sigma}{\sigma-1}}}$$

but such that

$$\sup_{\alpha\in\mathbb{N}^d} \frac{|H(f,\alpha)|}{r^{-|\alpha|}(\alpha!)^{-b}} = \infty, \qquad \forall b > \frac{1}{d}\frac{1}{2\sigma}$$

for all $r > 0$.

Thank you for your attention!

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