Sequence space representations for translation-modulation invariant function and distribution spaces

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Introduction

Introduction – Classical function spaces

Consider the following classical Schwartz function spaces

- For $1 \leq p < \infty$,

$$\mathcal{D}_{L^p} = \left\{ f \in C^\infty(\mathbb{R}^n) \mid f^{(\alpha)} \in L^p(\mathbb{R}^n),\ \forall \alpha \in \mathbb{N}^n \right\}$$

- The space

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$$\mathcal{S}(\mathbb{R}^n) = \{\varphi \in C^\infty(\mathbb{R}^n) \mid \sup_{x\in\mathbb{R}^n} (1+|x|)^k |\varphi^{(\alpha)}(x)| < \infty,\ \forall k \in \mathbb{N}, \alpha \in \mathbb{N}^n\}$$

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$$\mathcal{D}'_{L^1} = (\dot{\mathcal{B}})' = \left\{f \in \mathcal{S}'(\mathbb{R}^n) \mid f * \varphi \in L^1(\mathbb{R}^n),\ \forall \varphi \in \mathcal{S}(\mathbb{R}^n)\right\}$$
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Consider for a lattice Λ and a Banach space E the sequence spaces

$$s(\Lambda; E) = \{(e_\lambda)_{\lambda\in\Lambda} \in E^\Lambda \mid (1+|\lambda|)^k \|e_\lambda\|_E < \infty, \text{ for all } k \in \mathbb{N}\}$$
$$s'(\Lambda; E) = \{(e_\lambda)_{\lambda\in\Lambda} \in E^\Lambda \mid (1+|\lambda|)^{-k} \|e_\lambda\|_E < \infty, \text{ for some } k \in \mathbb{N}\}$$

If $\Lambda = \mathbb{Z}^n$ or $E = \mathbb{C}$ we leave it out of the respective notation. In particular, we write $s = s(\mathbb{Z}^n)$ and $s' = s'(\mathbb{Z}^n)$.

For any lattice Λ we have that $s(\Lambda) \cong s$ and $s'(\Lambda) \cong s'$. Moreover, we have the isomorphisms

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We have the following sequence space representations:

- In the 1980s, Vogt and Valdivia independently showed:

$$\mathcal{D}_{L^p} \cong \ell^p \widehat{\otimes} s, \qquad \mathcal{B} \cong \ell^\infty \widehat{\otimes} s, \qquad \dot{\mathcal{B}} \cong c_0 \widehat{\otimes} s,$$

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Introduction – The spaces $\mathcal{D}_E$ and $\mathcal{D}'_E$

Let $E \subseteq \mathcal{S}'(\mathbb{R}^n)$ be a Banach space.

- We define the test function space

$$\mathcal{D}_E = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) \mid f^{(\alpha)} \in E,\ \forall \alpha \in \mathbb{N}^n \right\}$$

topologized via the norms

$$\|f\|_{E,k} = \sup_{|\alpha| \leq k} \|f^{(\alpha)}\|_E, \qquad k \in \mathbb{N}$$

- We define the distribution space

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Introduction – Goal

In this talk:

- We will provide sequence space representations for $\mathcal{D}_E$ and $\mathcal{D}'_E$ for a suitable class of Banach spaces containing $L^p(\mathbb{R}^n)$ and $C_0(\mathbb{R}^n)$.
- We will discuss some of its topological consequences.
- We will provide an isomorphic classification of the classical spaces $\mathcal{D}_{L^p}, \dot{\mathcal{B}}, \mathcal{B}$, and the mixed spaces $\mathcal{D}_{L^{p_1,p_2}}$ and $\mathcal{D}_{L^{p_1}\widehat{\otimes}_\pi L^{p_2}}$.

This talk is based on the preprint

A. Debrouwere, L. Neyt, *Sequence space representations for translation-modulation invariant function and distribution spaces*, arXiv:2111.00334.

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Translation-modulation invariant Banach space of distributions

TMIB – Solid Banach spaces

A Banach space E is called solid if

$$\forall f \in E, g \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \ : \ |g| \leq |f| \text{ a.e.} \implies g \in E \text{ and } \|g\|_E \ \leq \|f\|_E$$

To a solid Banach space E and lattice Λ we may naturally associate a discrete space

$$E_d(\Lambda) = \left\{ c \in \mathbb{C}^\Lambda \mid \sum_{\lambda \in \Lambda} c_\lambda 1_{\lambda + U} \in E \right\}$$

with U such that $\{\lambda + U \mid \lambda \in \Lambda\}$ are pairwise disjoint.

We have

$$(L^p(\mathbb{R}^n))_d(\Lambda) = \ell^p(\Lambda), \qquad p \in [1, \infty]$$

There are some problems:

- Solid Banach spaces cannot detect continuity, e.g., $C_0(\mathbb{R}^n)$ is **not** solid.
- In general, they are not closed under completed tensor products, e.g., $L^p(\mathbb{R}^n) \widehat{\otimes}_\pi L^p(\mathbb{R}^n)$ is not solid for $1 < p \leq 2$ (2019, Dimovski, Pilipović, Prangoski, Vindas)

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$$(L^p(\mathbb{R}^n))_d(\Lambda) = \ell^p(\Lambda), \qquad p \in [1, \infty]$$

There are some problems:

- Solid Banach spaces cannot detect continuity, e.g., $C_0(\mathbb{R}^n)$ is **not** solid.
- In general, they are not closed under completed tensor products, e.g., $L^p(\mathbb{R}^n) \widehat{\otimes}_\pi L^p(\mathbb{R}^n)$ is not solid for $1 < p \leq 2$ (2019, Dimovski, Pilipović, Prangoski, Vindas)

TMIB – Definition (1)

Let $T_x f = f(\ \cdot - x)$ and $M_\xi f = e^{2\pi i \xi \cdot} f(\cdot)$ denote the translation and modulation operators, respectively.

Definition

A Banach space E is called a translation-modulation invariant Banach space of distributions (TMIB) if it satisfies:

(*i*) the continuous dense inclusions $\mathcal{S}(\mathbb{R}^n) \hookrightarrow E \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ hold;

(*ii*) $T_x(E) \subseteq E$ and $M_\xi(E) \subseteq E$ for all $x, \xi \in \mathbb{R}^d$;

(*iii*) There exists $C, q > 0$ such that

$$\|T_x\|_{\mathcal{L}(E)} \leq C(1+|x|)^q \quad \text{and} \quad \|M_\xi\|_{\mathcal{L}(E)} \leq C(1+|\xi|)^q$$

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TMIB – Definition (2)

Definition

If $E \cong (E_0)'_b$, E_0 a TMIB, then E is called a dual translation-modulation invariant Banach space of distributions (DTMIB).

In particular, a DTMIB E satisfies

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TMIB – Examples (1)

- A measurable weight function $w : \mathbb{R}^n \to (0, \infty)$ is called *polynomially bounded* if there exist $C, q > 0$

$$w(x + y) \leq C w(x)(1 + |y|)^q, \qquad \forall x, y \in \mathbb{R}^n$$

For a polynomially bounded weight function w and a solid (D)TMIB, we consider the space

$$E_w = \{f \in L^1_{\text{loc}}(\mathbb{R}^n) \mid f \cdot w \in E\}$$

Then E_w is again a solid (D)TMIB.
In particular we find

$$L^p_w(\mathbb{R}^n) = \left\{f \in L^1_{\text{loc}}(\mathbb{R}^n) \mid f \cdot w \in L^p(\mathbb{R}^n)\right\}$$

which is a solid TMIB if $1 \leq p < \infty$, and a solid DTMIB if $p = \infty$.
We may also consider

$$C_{0,w}(\mathbb{R}^n) = \left\{f \in C(\mathbb{R}^n) \mid \lim_{|x| \to \infty} |f(x)| w(x) = 0\right\}$$

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TMIB – Examples (2)

- For $p_1, p_2 \in [1, \infty]$, the mixed norm space $L^{p_1,p_2}(\mathbb{R}^{2n})$ is a solid TMIB if $1 \leq p_1, p_2 < \infty$ and a solid DTMIB if $1 < p_1, p_2 \leq \infty$.

  In particular, for $p_1, p_2 \in [1, \infty]$ and any polynomially bounded weight function $w : \mathbb{R}^n \to (0, \infty)$, the space $L^{p_1,p_2}_w(\mathbb{R}^{2n})$ of all measurable functions f on $\mathbb{R}^{2n}$ such that

$$\left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x,\xi)| w(x,\xi) dx \right)^{p_2/p_1} d\xi \right)^{1/p_2} < \infty$$

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TMIB – Examples (3)

- If E is a (D)TMIB, then

$$\mathcal{F}E = \{f \in \mathcal{S}'(\mathbb{R}^n) \mid \mathcal{F}^{-1}f \in E\}$$

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$$\mathcal{F}L^p_w(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) \mid \|\mathcal{F}^{-1}(f)(\xi)w(\xi)\|_{L^p} < \infty\}$$

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- If E_1 and E_2 are TMIB, then so are

$$E_1 \widehat{\otimes}_\pi E_2 \quad \text{and} \quad E_1 \widehat{\otimes}_\varepsilon E_2$$

In particular, for any $1 \leq p_1, p_2 < \infty$,

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TMIB – Discrete space

Let $\chi \in \mathcal{D}(U) \setminus \{0\}$, we define

$$E_d(\Lambda) = E_{d,\chi}(\Lambda) := \left\{ c \in \mathbb{C}^\Lambda \mid S_\chi(c) := \sum_{\lambda \in \Lambda} c_\lambda T_\lambda \chi \in E \right\}$$

- $E_d(\Lambda)$ is a Banach space with the norm $\|c\|_{E_d(\Lambda)} = \|c\|_{E_{d,\chi}(\Lambda)} := \|S_\chi(c)\|_E$.
- $E_d(\Lambda)$ does not depend on the choice of $\chi \in \mathcal{D}(U) \setminus \{0\}$.
- If E is solid the definition of $E_d(\Lambda)$ coincides with the definition for solid Banach spaces, i.e.

$$E_d(\Lambda) = \left\{ c \in \mathbb{C}^\Lambda \mid \sum_{\lambda \in \Lambda} c_\lambda 1_{\lambda + U} \in E \right\}$$

- For a TMIB E we have the dense inclusions

$$s(\Lambda) \hookrightarrow E_d(\Lambda) \hookrightarrow s'(\Lambda)$$

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- Let E be a solid (D)TMIB and w a polynomially bounded weight function. We have

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In particular, for $p \in [1,\infty]$ we have

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Similarly, one has

$$(C_{0,w}(\mathbb{R}^n))_d(\Lambda) = c_{0,w}(\Lambda) = \{c \in \mathbb{C}^\Lambda \mid \lim_{|\lambda|\to\infty} |c_\lambda w(\lambda)| = 0\}$$

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$$(E_w)_d(\Lambda) = \left\{c \in \mathbb{C}^\Lambda \mid (c_\lambda w(\lambda))_{\lambda\in\Lambda} \in E_d(\Lambda)\right\}$$

In particular, for $p \in [1,\infty]$ we have

$$(L^p_w(\mathbb{R}^n))_d(\Lambda) = \ell^p_w(\Lambda) = \{c \in \mathbb{C}^\Lambda \mid \|(c_\lambda w(\lambda))_{\lambda\in\Lambda}\|_{\ell^p} < \infty\}$$

For $p_1, p_2 \in [1,\infty]$ with $\{p_1,p_2\} \neq \{1,\infty\}$

$$(L^{p_1,p_2}_w(\mathbb{R}^{2n}))_d(\Lambda) = \ell^{p_1,p_2}_w(\Lambda) = \{c \in \mathbb{C}^\Lambda \mid \|(c_\lambda w(\lambda))_{\lambda\in\Lambda}\|_{\ell^{p_1,p_2}} < \infty\}$$

Similarly, one has

$$(C_{0,w}(\mathbb{R}^n))_d(\Lambda) = c_{0,w}(\Lambda) = \{c \in \mathbb{C}^\Lambda \mid \lim_{|\lambda|\to\infty} |c_\lambda w(\lambda)| = 0\}$$

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TMIB – Discrete space examples (2)

- Let E be a solid (D)TMIB and $\Lambda^\perp$ the dual lattice of Λ. Define

$$E(\mathbb{R}^n/\Lambda^\perp) = \{f \in L^1_{\text{loc}} \mid f \text{ is } \Lambda^\perp\text{-periodic and } f1_K \in E \text{ for all } K \Subset \mathbb{R}^n \text{ compact } \}$$

Then

$$E(\mathbb{R}^n/\Lambda^\perp) \to (\mathcal{F}E)_d(\Lambda), \quad f \mapsto \left(\frac{1}{\text{vol}(\Lambda^\perp)} \int_{I_{\Lambda^\perp}} f(x) e^{-2\pi i \lambda \cdot x} dx \right)_{\lambda \in \Lambda}$$

is a topological isomorphism.

For example, for $p \in [1, \infty]$ we have

$$(\mathcal{F}L^p(\mathbb{R}^n))_d(\mathbb{Z}^n) \cong b_p = \left\{ c \in \mathbb{C}^{\mathbb{Z}^n} \mid \sum_{k \in \mathbb{Z}^n} c_k e^{2\pi i k \cdot x} \in L^p(\mathbb{R}^n/\mathbb{Z}^n) \right\}$$

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TMIB – Discrete space examples (3)

- Let E_j be a TMIB and Λ_j be a lattice on $\mathbb{R}^{n_j}$, $j = 1, 2$, then,

$$(E_1 \widehat{\otimes}_\tau E_2)_d(\Lambda_1 \times \Lambda_2) \cong (E_1)_d(\Lambda_1) \widehat{\otimes}_\tau (E_2)_d(\Lambda_2), \qquad \tau = \varepsilon \text{ or } \pi$$

In particular, for $1 \leq p_1, p_2 < \infty$ we have

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TMIB – Discrete space (3)

Proposition (Debrouwere, Prangoski)

- The bilinear mapping

$$E_d(\Lambda) \times \mathcal{S}(\mathbb{R}^n) \to E, \quad (c, \varphi) \mapsto S_\varphi(c) := \sum_{\lambda \in \Lambda} c_\lambda \, T_\lambda \varphi$$

  is well-defined and continuous.

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Gabor frames

Gabor frames – Dual windows (1)

Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ and $\Lambda = \Lambda_0 \times \Lambda_1$.

We consider the analysis operator

$$C_{\psi,\Lambda} : \mathcal{S}'(\mathbb{R}^n) \to s'(\Lambda) : \quad f \mapsto (\langle f, \overline{M_{\lambda_1} T_{\lambda_0} \psi} \rangle)_{\lambda \in \Lambda}$$

and the synthesis operator

$$D_{\psi,\Lambda} : s'(\Lambda) \to \mathcal{S}'(\mathbb{R}^n) : \quad c \mapsto \sum_{\lambda \in \Lambda} c_\lambda M_{\lambda_1} T_{\lambda_0} \psi$$

Definition

Two windows $\psi, \gamma \in \mathcal{S}(\mathbb{R}^n)$ are called a pair of dual windows on Λ if

$$D_{\gamma,\Lambda} \circ C_{\psi,\Lambda} = \mathrm{id}_{\mathcal{S}'(\mathbb{R}^n)}$$

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Gabor frames – Dual windows (2)

Wexler-Raz biorthogonality relations

For $a, b > 0$, ψ and γ are a pair of dual windows on $a\mathbb{Z}^n \times b\mathbb{Z}^n$ if and only if

$$\frac{1}{(ab)^n} C_{\psi, \frac{1}{a}\mathbb{Z}^n \times \frac{1}{b}\mathbb{Z}^n} \circ D_{\gamma, \frac{1}{a}\mathbb{Z}^n \times \frac{1}{b}\mathbb{Z}^n} = \mathrm{id}_{s'(\frac{1}{a}\mathbb{Z}^n \times \frac{1}{b}\mathbb{Z}^n)}$$

Gabor frames – Definition

Given a window $\psi \in \mathcal{S}(\mathbb{R}^n)$, the set of time-frequency shifts

$$\mathcal{G}(\Lambda, \psi) = \{M_{\lambda_1} T_{\lambda_0} \psi \mid \lambda \in \Lambda\}$$

is called a Gabor frame on Λ if there are $A, B > 0$ such that

$$A\|f\|_{L^2} \leq \|((f, \varphi)_{L^2})_{\varphi \in \mathcal{G}(\Lambda,\psi)}\|_{\ell^2(\Lambda)} \leq B\|f\|_{L^2}, \qquad \forall f \in L^2(\mathbb{R}^n)$$

Theorem (1992, Walnut)

For any $\psi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ there exist $a, b > 0$ such that ψ is a Gabor frame on $a\mathbb{Z}^n \times b\mathbb{Z}^n$.

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Gabor frames – Strategy (1)

To show the sequence space representations, we will employ the following strategy.

(1) Let $X \subseteq \mathcal{S}'(\mathbb{R}^d)$ be the space for which we wish to find the representation and suppose that for any lattice $\Lambda = \Lambda_0 \times \Lambda_1$ we have a corresponding sequence space $Y_\Lambda \subseteq s'(\Lambda)$.

(2) For any $\psi, \gamma \in \mathcal{S}(\mathbb{R}^n)$ we show that

$$C_\psi : X \to Y_\Lambda \quad \text{and} \quad D_\gamma : Y_\Lambda \to X$$

are well-defined and continuous.

(3) Then, if ψ and γ are a pair of dual windows on $a\mathbb{Z}^n \times b\mathbb{Z}^n$ for certain $a, b > 0$ and if $Y \cong Y_{a\mathbb{Z}^n \times b\mathbb{Z}^n} \cong Y_{\frac{1}{a}\mathbb{Z}^n \times \frac{1}{b}\mathbb{Z}^n}$, we have

$$X \underset{c}{\subset} Y \quad \text{and} \quad Y \underset{c}{\subset} X$$

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Gabor frames – Strategy (2)

Pełczyński decomposition method

If X and Y_0 are locally convex Hausdorff spaces such that

$$X \underset{c}{\subset} Y_0 \widehat{\otimes} s \quad \text{and} \quad Y_0 \widehat{\otimes} s \underset{c}{\subset} X$$

then

$$X \cong Y_0 \widehat{\otimes} s$$

The same holds true for s exchanged with s'.

(4) If then $Y \cong Y_0 \widehat{\otimes} s$ or $Y \cong Y_0 \widehat{\otimes} s'$ for some locally convex Hausdorff space Y_0, we may conclude

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Gabor frames – Characterizations of $\mathcal{D}_E$ and $\mathcal{D}'_E$

Proposition (Debrouwere, N.)

For any $\psi \in \mathcal{S}(\mathbb{R}^n)$, the following mappings are well-defined and continuous:

- $C_{\psi,\Lambda} : \mathcal{D}_E \to s(\Lambda_1; E_d(\Lambda_0))$ and $D_{\psi,\Lambda} : s(\Lambda_1; E_d(\Lambda_0)) \to \mathcal{D}_E$
- $C_{\psi,\Lambda} : \mathcal{D}'_E \to s'(\Lambda_1; E_d(\Lambda_0))$ and $D_{\psi,\Lambda} : s'(\Lambda_1; E_d(\Lambda_0)) \to \mathcal{D}'_E$

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Main results

Main results – The sequence space representations

Theorem (Debrouwere, N.)

$$\mathcal{D}_E \cong E_d \widehat{\otimes} s \quad \text{and} \quad \mathcal{D}'_E \cong E_d \widehat{\otimes} s'$$

Proof: By applying our strategy with the Gaussian window

$$\psi(x) = e^{-\pi |x|^2} \in \mathcal{S}(\mathbb{R}^n)$$

for which $\mathcal{G}(\mathbb{Z}^n \times b\mathbb{Z}^n, \psi)$ is a Gabor frame for any $0 < b < 1$ and noting that for any $c > 0$

$$s(c\mathbb{Z}^n; E_d(\mathbb{Z}^n)) \cong E_d \widehat{\otimes} s(c\mathbb{Z}^n) \cong E_d \widehat{\otimes} s$$

and

$$s'(c\mathbb{Z}^n; E_d(\mathbb{Z}^n)) \cong E_d \widehat{\otimes} s'(c\mathbb{Z}^n) \cong E_d \widehat{\otimes} s'$$

Main results – Corollaries (1)

Corollaries

(1) The space $\mathcal{D}_E$ is a quasinormable Fréchet space.

(2) The space $\mathcal{D}'_E$ is a complete (LB)-space, hence ultrabornological.

(3) Both the spaces $\mathcal{D}_E$ and $\mathcal{D}'_E$ are **not** Montel.

(4) Suppose that E is a TMIB. Then

$$(\mathcal{D}_E)' = \mathcal{D}'_{E'}$$

as locally convex spaces.

(5) We have the following equalities

$$\mathcal{D}'_{L^p_w} = (\mathcal{D}_{L^q_{1/w}})', \qquad \mathcal{B}'_w = (\mathcal{D}_{L^1_{1/w}})',$$

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$$(\mathcal{D}_E)' = \mathcal{D}'_{E'}$$

as locally convex spaces.

(5) We have the following equalities

$$\mathcal{D}'_{L^p_w} = (\mathcal{D}_{L^q_{1/w}})', \qquad \mathcal{B}'_w = (\mathcal{D}_{L^1_{1/w}})',$$

$$\mathcal{D}'_{L^1_w} = (\dot{\mathcal{B}}_{1/w})', \qquad \dot{\mathcal{B}}'_w = \overline{\mathcal{D}(\mathbb{R}^n)}^{\mathcal{B}'_w}$$

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Main results – Corollaries (1)

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(1) The space $\mathcal{D}_E$ is a quasinormable Fréchet space.

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Main results – Corollaries (2)

Analogous results to (1), (2), (4) and (5) have been shown for the ultradifferentiable case in:

L. Neyt, *Convolutors of translation-modulation invariant Banach spaces of ultradistributions*, J. Math. Anal. Appl. **507** (2022), 125759.

Main results – Corollaries (3)

Let E be a solid (D)TMIB. We define

$$\mathcal{F}E_\infty = \varprojlim_{m\to\infty} \mathcal{F}(E_{(1+|\cdot|)^m}) \quad \text{and} \quad \mathcal{F}E_{-\infty} = \varinjlim_{m\to\infty} \mathcal{F}(E_{(1+|\cdot|)^{-m}})$$

Corollary

We have

$$\mathcal{F}E_\infty = \mathcal{D}_{\mathcal{F}E} \cong (\mathcal{F}E)_d\widehat{\otimes}s \quad \text{and} \quad \mathcal{F}E_{-\infty} = \mathcal{D}'_{\mathcal{F}E} \cong (\mathcal{F}E)_d\widehat{\otimes}s'$$

In particular, for $p \in [1,\infty]$ we have for the Sobolev type spaces

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Isomorphic classification

Isomorphic classification – Sequence space representations

We find the following sequence space representations

$$
\begin{array}{lcll}
\mathcal{D}_{L^p} & \cong & \ell^p \widehat{\otimes} s & p \in [1, \infty) \\
\mathcal{B} & \cong & \ell^\infty \widehat{\otimes} s & \\
\dot{\mathcal{B}} & \cong & c_0 \widehat{\otimes} s & \\
\mathcal{D}_{L^{p_1, p_2}} & \cong & \ell^{p_1, p_2} \widehat{\otimes} s & p_1, p_2 \in [1, \infty) \\
\mathcal{D}_{L^{p_1} \widehat{\otimes}_\pi L^{p_2}} & \cong & \ell^{p_1} \widehat{\otimes}_\pi \ell^{p_2} \widehat{\otimes}_\pi s & p_1, p_2 \in [1, \infty)
\end{array}
$$

Isomorphic classification – Lemmas (1)

Lemma

Let X and Y be Banach space. Suppose $\ell^p(X) \cong X$ for some $p \in [1, \infty)$ or $c_0(X) \cong X$.

(*i*) We have

$$s(Y) \underset{c}{\subset} s(X) \Longrightarrow Y \underset{c}{\subset} X$$

(*ii*) Suppose additionally that $\ell^p(Y) \cong Y$ for some $p \in [1, \infty)$ or $c_0(Y) \cong Y$. Then

$$s(X) \cong s(Y) \Longrightarrow X \cong Y$$

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Isomorphic classification – Lemmas (2)

Lemma

Let $1 < p_1, p_2 < \infty$ and put

$$t(p_1, p_2) = \frac{1}{\min(1, \frac{1}{p_1} + \frac{1}{p_2})}$$

(*i*) $\ell^{p_1} \widehat{\otimes}_\pi \ell^{p_2}$ is not isomorphic to a complemented subspace of a Banach space with an unconditional basis (Gorden, Lewis, '74)

(*ii*) $\ell^{t(p_1,p_2)}(\ell^{p_1} \widehat{\otimes}_\pi \ell^{p_2}) \cong \ell^{p_1} \widehat{\otimes}_\pi \ell^{p_2}$ (Arias, Farmer, '96)

(*iii*) Let $1 \leq p < \infty$. Then, ℓ^p is isomorphic to a closed subspace of $\ell^{p_1} \widehat{\otimes}_\pi \ell^{p_2}$ if and only if $p \in \{p_1, p_2, t(p_1, p_2)\}$ (Arias, Farmer, '96)

Isomorphic classification – First result

Theorem (Debrouwere, N.)

Let $1 < p_1, p_2 < \infty$.

(*i*) $\mathcal{D}_{L^{p_1}\widehat{\otimes}_\pi L^{p_2}}$ is **not** isomorphic to a complemented subspace of $\mathcal{D}_{L^{q_1,q_2}}$, $1 \leq q_1, q_2 < \infty$, $\dot{\mathcal{B}}$, or $\mathcal{B}$.

(*ii*) Let $1 < q_1, q_2 < \infty$, then,

$$\mathcal{D}_{L^{p_1}\widehat{\otimes}_\pi L^{p_2}} \cong \mathcal{D}_{L^{q_1}\widehat{\otimes}_\pi L^{q_2}} \iff \{p_1, p_2\} = \{q_1, q_2\}$$

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Isomorphic classification – Second result

Theorem (Debrouwere, N.)

(*i*) None of the space $\mathcal{D}_{L^p}$, $1 \leq p < \infty$, $\dot{\mathcal{B}}$, $\mathcal{B}$ is isomorphic to a complemented subspace of any other of these spaces.

(*ii*) The spaces $\mathcal{D}_{L^{p_1,p_2}}$, $1 \leq p_1, p_2 < \infty$, are mutually non-isomorphic.

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