Sequence space representations for ultradifferentiable functions: explicit and implicit approaches

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Absolute Schauder bases

A locally convex space E is said to have an absolute Schauder basis $(e_n)_{n\in\mathbb{N}}$ if there exist linear mappings $\lambda_n : E \to \mathbb{C}$ such that, for each $e \in E$,

$$e = \sum_{n\in\mathbb{N}} \lambda_n(e) e_n,$$

and, for each continuous seminorm p on E there exists a continuous seminorm q such that

$$\sum_{n\in\mathbb{N}} |\lambda_n(e)| p(e_n) \leq q(e).$$

Schauder bases allow for a clear determination of topological invariants and continuous linear operators, as well as the applicability of functional analytic machinery (e.g., splitting theorems).

Absolute Schauder bases for Fréchet spaces

Let E be a Fréchet space (= complete metrizable TVS) with increasing fundamental sequence of norms $(\|\cdot\|_r)_{r\in\mathbb{N}}$.

Suppose E has an absolute Schauder basis $(e_n)_{n\in\mathbb{N}}$.

Put

$$a_{r_0,n} := \|e_n\|_{r_0} \leq \|e_n\|_{r_1} =: a_{r_1,n}, \qquad n \in \mathbb{N},\ r_0 \leq r_1,$$

then

$$E \cong \bigcap_{r\in\mathbb{N}} \left\{ (c_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \mid \sum_{n\in\mathbb{N}} |c_n| a_{r,n} < \infty \right\}$$

Remark: A similar analysis can also be done for (LB)-spaces (= inductive limits of Banach spaces).

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Gelfand-Shilov spaces

For $\mu, \tau > 0$ we define the Gelfand-Shilov space of Beurling type Σ_τ^μ as all those functions $f \in C^\infty(\mathbb{R})$ such that

$$\forall k > 0 : \quad \sup_{p,q \in \mathbb{N}} \sup_{x \in \mathbb{R}} \frac{x^q |f^{(p)}(x)|}{k^{p+q}(p!)^\mu (q!)^\tau} < \infty$$

Equivalently [Chung, Chung, Kim, 1996], Σ_τ^μ are all those $f \in L^2(\mathbb{R})$ for which

$$\forall h > 0 : \quad |f(x)| \lesssim e^{-h|x|^{\frac{1}{\tau}}} \quad \text{and} \quad |\widehat{f}(\xi)| \lesssim e^{-h|\xi|^{\frac{1}{\mu}}}$$

We have [Gel'fand, Shilov, 1968]

$$\Sigma_\tau^\mu \neq \{0\} \quad \Longleftrightarrow \quad \mu + \tau > 1$$

Remark: Our results also cover spaces of more general time-frequency decay (so-called Beurling-Björck spaces).

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Known sequence space representations

For the following spaces representations are known:

- Schwartz, in 1950, showed that $\mathcal{S}$ is isomorphic to

$$s = \{c = (c_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \mid \forall r > 0 : \sup_{n\in\mathbb{N}} (1+n)^r |c_n| < \infty\}$$

  using Hermite series expansions.
- Fourier invariant Gelfand-Shilov spaces: Using Hermite series expansions, sequence space representations were established for
  * Σ^{μ}_{μ}, $\mu > 1/2$ [Pilipović, 1986].
  * $\mathcal{S}^{(M)}_{(M)}$ [Langenbruch, 2006]; further generalizations in [Boiti, Jornet, Oliaro, Schindl, 2020].
- Σ^{μ}_{τ} when $\mu/\tau \in \mathbb{Q}$, using eigenfunction expansions with respect to certain anisotropic Shubin operators [Cappiello, Gramchev, Pilipović, Rodino, 2019].
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Our results

In this talk, we demonstrate two ways of obtaining sequence space representations:

- Explicitely, using a Wilson basis.
  $\hookrightarrow$ Used in [Bargetz, Debrouwere, Nigsch, 2022] to establish commutative and explicit sequence space representations for all spaces appearing in Schwartz's distribution theory.
- Implicitely, by the use of Gabor frames and Pełczynski decomposition methods.
  $\hookrightarrow$ First used in [Debrouwere, N., 2022].

Both approaches impose time-frequency bounds on the Gelfand-Shilov spaces due to the specific frames being used.

This talk is based on our recent preprint:

A. Debrouwere, L. Neyt, *Sequence space representations of Beurling-Björck spaces via Gabor frames and Wilson bases*, arXiv: 2508.05565.

In an upcoming work, we establish sequence space representations for $\mathcal{O}_M(\Sigma_\tau^\mu)$ using similar techniques.

Gabor frames – Definition

Let $\psi \in L^2(\mathbb{R})$ and $a, b > 0$. The sets of time-frequency shifts

$$\mathcal{G}(\psi, a, b) = \{T_{ak} M_{bn} \psi = e^{2\pi i bn \cdot} \psi(\cdot - ak) \mid (k, n) \in \mathbb{Z}^2\}$$

is called a Gabor frame if, for some $A, B > 0$,

$$A\|f\|_{L^2}^2 \leq \sum_{(k,n)\in\mathbb{Z}^2} |(f, T_{ak} M_{bn} \psi)_{L^2}|^2 \leq B\|f\|_{L^2}^2$$

For $\psi \in \mathcal{S}$, we consider the Gabor analysis operator

$$C_\psi^{a,b} : L^2(\mathbb{R}) \to \ell^2(\mathbb{Z}^2), \quad f \mapsto ((f, T_{ak} M_{bn} \psi)_{L^2})_{(k,n)\in\mathbb{Z}^2}$$

and Gabor synthesis operator

$$D_\psi^{a,b} : \ell^2(\mathbb{Z}^2) \to L^2(\mathbb{R}), \quad (c_{k,n})_{(k,n)\in\mathbb{Z}^2} \mapsto \sum_{(k,n)\in\mathbb{Z}^2} c_{k,n} T_{ak} M_{bn} \psi$$

which are continuous.

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Gabor frames – Pairs of dual windows

By the Balian-Low theorem

$$\psi \in \mathcal{S} \quad \Longrightarrow \quad \mathcal{G}(\psi, a, b) \text{ is not an orthonormal basis}$$

For $\psi, \gamma \in \mathcal{S}$ and $a, b > 0$ we call (ψ, γ) a pair of dual windows (on $a\mathbb{Z} \times b\mathbb{Z}$) if

$$D_{\gamma}^{a,b} \circ C_{\psi}^{a,b} = \mathrm{id}_{L^2(\mathbb{R})}$$

In this case, also (γ, ψ) is a pair of dual windows and $\mathcal{G}(\psi, a, b)$ and $\mathcal{G}(\gamma, a, b)$ are Gabor frames.

Consider the canonical window

$$\gamma^{\circ} = \left(D_{\psi}^{a,b} \circ C_{\psi}^{a,b} \right)^{-1} \psi$$

Then (ψ, γ°) is a pair of dual windows.

By the Wexler-Raz biorthogonality relations

$$(\psi, \gamma) \text{ is a pair of dual windows} \quad \Longleftrightarrow \quad \frac{1}{ab} C_{\psi}^{\frac{1}{b}, \frac{1}{a}} \circ D_{\gamma}^{\frac{1}{b}, \frac{1}{a}} = \mathrm{id}_{\ell^2(\mathbb{Z}^n)}$$

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Gabor frames – Duals windows in Gelfand-Shilov spaces

When $\mu \leq 1$ or $\tau \leq 1$ then

$$\psi \in \Sigma_\tau^\mu \quad \not\Longrightarrow \quad \gamma^\circ \in \Sigma_\tau^\mu$$

We call Σ_τ^μ Gabor accessible if there exist $\psi, \gamma \in \Sigma_\tau^\mu$ such that (ψ, γ) is a pair of dual windows on some lattice $a\mathbb{Z} \times b\mathbb{Z}$.

Two sufficient conditions for Gabor accessibility of Σ_τ^μ:

- $\min\{\mu, \tau\} > 1/2$: Gaussians have Gabor duals in $\mathcal{S}_{1/2}^{1/2}$ [Janssen, 1994].
- $\max\{\mu, \tau\} > 1$: Using compactly supported windows – or their Fourier transforms [Christensen, 2006].

Question: Is any non-trivial Σ_τ^μ Gabor accessible?

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We consider the Wilson analysis operator

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- $\min\{\mu, \tau\} > 1$: A Wilson basis with exponential time-frequency decay was constructed in [Daubechies, Jaffard, Journé, 1991].
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Gabor frame and Wilson basis characterizations of the Gelfand-Shilov spaces

Let $\psi \in \Sigma_\tau^\mu$.

We find the following mapping properties:

- Gabor frames

$$\varphi \in \Sigma_\tau^\mu \underset{D_\psi^{a,b}}{\overset{C_\psi^{a,b}}{\rightleftarrows}} \forall r > 0 : |c_{k,n}| \lesssim e^{-r[|k|^{\frac{1}{\tau}} + |n|^{\frac{1}{\mu}}]}$$

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$$\varphi \in \Sigma_\tau^\mu \underset{\widetilde{D}_\psi}{\overset{\widetilde{C}_\psi}{\rightleftarrows}} \forall r > 0 : |c_{k,n}| \lesssim e^{-r[|k|^{\frac{1}{\tau}} + n^{\frac{1}{\mu}}]}$$

Sequence spaces

For a sequence $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ of positive numbers increasing to ∞ we put

$$\Lambda_\infty(\alpha) = \bigcap_{r>0} \left\{ (c_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \mid \sup_{n\in\mathbb{N}} |c_n| e^{r\alpha_n} < \infty \right\}$$

endowed with its natural Fréchet space topology.

For two sequences $\alpha = (\alpha_k)_{k\in\mathbb{N}}$ and $\beta = (\beta_n)_{n\in\mathbb{N}}$ of positive numbers increasing to ∞ we put

$$\Lambda_\infty(\alpha,\beta) = \bigcap_{r>0} \left\{ (c_{k,n})_{k,n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \mid \sup_{n\in\mathbb{N}} |c_{k,n}| e^{r[\alpha_k+\beta_n]} < \infty \right\}$$

Sequence spaces

For a sequence $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ of positive numbers increasing to ∞ we put

$$\Lambda_\infty(\alpha) = \bigcap_{r>0} \left\{ (c_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \mid \sup_{n\in\mathbb{N}} |c_n| e^{r\alpha_n} < \infty \right\}$$

endowed with its natural Fréchet space topology.

For two sequences $\alpha = (\alpha_k)_{k\in\mathbb{N}}$ and $\beta = (\beta_n)_{n\in\mathbb{N}}$ of positive numbers increasing to ∞ we put

$$\Lambda_\infty(\alpha,\beta) = \bigcap_{r>0} \left\{ (c_{k,n})_{k,n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \mid \sup_{n\in\mathbb{N}} |c_{k,n}| e^{r[\alpha_k+\beta_n]} < \infty \right\}$$

Explicit sequence space representations (1)

Theorem (Debrouwere, N.)

There exists a topological isomorphism

$$\Psi \colon \mathcal{S} \to \Lambda_\infty(\log(1+k), \log(1+n))$$

such that, for all $\mu, \tau > 1$,

$$\Psi_{|\Sigma^\mu_\tau} \colon \Sigma^\mu_\tau \to \Lambda_\infty(k^{\frac{1}{\mu}}, n^{\frac{1}{\tau}})$$

are topological isomorphisms.

Explicit sequence space representations (2)

Theorem (Debrouwere, N.)

Let $\mu_0 > 1$.

There exists a topological isomorphism

$$\Psi \colon \mathcal{S} \to \Lambda_\infty(\log(1+k), \log(1+n))$$

such that, for all $\mu \geq \mu_0$ and $\tau > 0$,

$$\Psi_{|\Sigma^\mu_\tau} \colon \Sigma^\mu_\tau \to \Lambda_\infty(k^{\frac{1}{\mu}}, n^{\frac{1}{\tau}})$$

are topological isomorphisms.

The Pełczynski-Vogt decomposition method

Theorem (Vogt, 1982)

Let $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ be a sequence increasing to ∞ such that

(*i*) $\log n = O(\alpha_n)$ (resp. $\log n = o(\alpha_n)$) [α is nuclear]

(*ii*) $\alpha_{2n} = O(\alpha_n)$ [α is stable]

Let E be a Fréchet space such that:

- E is isomorphic to a complemented subspace of $\Lambda_\infty(\alpha)$;
- $\Lambda_\infty(\alpha)$ is isomorphic to a complemented subspace of E.

Then

$$E \cong \Lambda_\infty(\alpha)$$

Implicit sequence space representations

Theorem (Debrouwere, N.)

For $\mu, \tau > 1/2$, it holds that

$$\Sigma_\tau^\mu \cong \Lambda_\infty(k^{\frac{1}{\mu}}, n^{\frac{1}{\tau}}) \cong \Lambda_\infty(n^{\frac{1}{\mu+\tau}}).$$

Isomorphic classification of the Gelfand-Shilov spaces

Theorem (Debrouwere, N.)

Let $\mu_i, \tau_i > 0$, $i = 1, 2$, be such that

$$\min\{\mu_i, \tau_i\} > 1/2 \quad \text{or} \quad \max\{\mu_i, \tau_i\} > 1, \qquad i \in \{1, 2\}$$

Then

$$\Sigma_{\tau_1}^{\mu_1} \cong \Sigma_{\tau_2}^{\mu_2}$$

if and only if

$$\mu_1 + \tau_1 = \mu_2 + \tau_2$$

Remark: We have $\Sigma_{\tau_1}^{\mu_1} = \Sigma_{\tau_2}^{\mu_2}$ as sets if and only if $\mu_1 = \mu_2$ and $\tau_1 = \tau_2$ (when spaces are non-trivial).

Proof of Pełczynski-Vogt decomposition (1)

Let Λ be a Fréchet space for which

1. $\Lambda \times \Lambda \cong \Lambda$;
2. There exists an exact sequence

$$0 \longrightarrow \Lambda \longrightarrow \Lambda \longrightarrow \Lambda^{\mathbb{N}} \longrightarrow 0$$

3. Each exact sequence of the form

$$0 \longrightarrow \Lambda \longrightarrow Z \longrightarrow \Lambda \longrightarrow 0$$

splits, i.e., $Z \cong \Lambda \times \Lambda$ $(\cong \Lambda)$

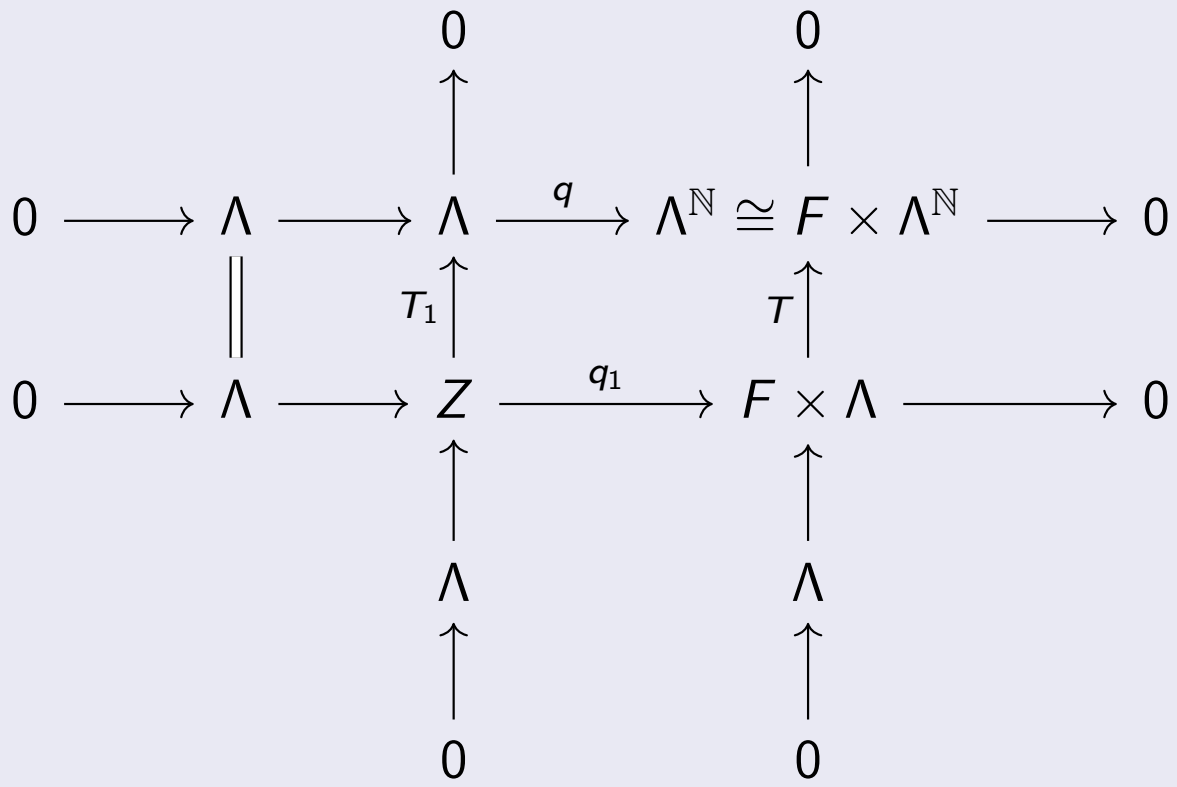
Then, the conclusion of the Pełczynski-Vogt decomposition holds:
If a Fréchet space E is isomorphic to a complemented subspace of Λ and vice-versa, then

$$E \cong \Lambda$$

Proof of Pełczynski-Vogt decomposition (2)

Lemma

For any complemented subspace F of Λ we have $F \times \Lambda \cong \Lambda$.



Proof of Pełczynski-Vogt decomposition (3)

Suppose Λ is isomorphic to a complemented subspace of E. Then

$$E \cong F \times \Lambda$$

for some Fréchet space F. Then F is also isomorphic to a complemented subspace of E.

Therefore, if E is isomorphic to a complemented subspace of Λ, so is F.

By the previous lemma

$$E \cong F \times \Lambda \cong \Lambda$$

Thank you for your attention!

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