On the existance of continuous right inverses for surjective operators and its applications

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Short exact sequences and splitting

Introduction

Suppose we have a surjective continuous linear operator

$$Q : W \to Z.$$

Question: When does Q have a continuous linear right inverse?
That is, when does there exist a continuous linear map $R : Z \to W$ such that $Q \circ R = \mathrm{id}_Z$?

Note: If such an R exists, the operator

$$P = R \circ Q$$

is a projection on W. Under some assumptions, we then have

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$$W \cong Z \times \ker Q.$$

Short exact sequences

We say that the complex

$$0 \longrightarrow E \xrightarrow{\iota} W \xrightarrow{Q} Z \longrightarrow 0$$

is a short exact sequence if

- $\iota : E \to W$ is a continuous linear injection
- $Q : W \to Z$ is a continuous linear surjection
- $\operatorname{Im} \iota = \ker Q$

We say that the complex is topologically exact if ι and Q are topological homomorphisms.

We say that the complex splits if Q has a continuous linear right inverse. When the complex is topologically exact, this is equivalent to ι having a continuous linear left inverse (sufficient when Q is open).

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Splitting for certain categories of locally convex spaces

- Banach spaces: No general conditions for splitting are known. However, the following type of short exact sequences always split:

$$0 \longrightarrow \ell^\infty(I) \longrightarrow W \longrightarrow Z \longrightarrow 0$$

and

$$0 \longrightarrow E \longrightarrow W \longrightarrow \ell^1(I) \longrightarrow 0$$

- Fréchet spaces: A thorough splitting theory was developed by Vogt et al. E.g., if E or Z is nuclear, E satisfies the condition (Ω)

$$\forall n \in \mathbb{N}\, \exists m \geq n\ \forall k \geq m\, \exists \theta \in (0,1)\, \forall e' \in E' : \quad \|e'\|_m^* \lesssim (\|e'\|_k^*)^\theta (\|e'\|_n^*)^{1-\theta}$$

and Z satisfies the condition (DN)

$$\exists n \in \mathbb{N}\, \forall m \geq n\, \exists k \geq m\, \forall z \in Z : \quad \|z\|_m^2 \lesssim \|z\|_k \|z\|_n$$

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Our work

Working via duality to obtain splitting results often imposes unnecessary topological conditions on the spaces involved.

Inspired by earlier work due to Vogt, we established a much more general splitting theory for (LB)-*spaces* in:

A. Debrouwere, L. Neyt, *An extension result for (LB)-spaces and the surjectivity of tensorized mappings*, Annali di Matematica **203** (2024), 1753–1781.

The content of this talk will be primarily based on this paper.

We also applied these results to the specific case of the *Stieltjes moment problem*, which we will consider at the end of the talk.

A. Debrouwere, L. Neyt, *The vector-valued Stieltjes moment problem with general exponents*, Banach J. Math. Anal. **18** (2024), Article number: 53.

Splitting in the category of (LB)-spaces

(LB)-spaces

Let $(Z_n)_{n\in\mathbb{N}}$ be Banach spaces and Z a vector space such that

$$Z_n \subseteq Z_m \subseteq Z, \qquad \forall n \leq m$$

and

$$Z = \bigcup_{n\in\mathbb{N}} Z_n$$

We say that Z is an (LB)-space if Z is endowed with the finest lcHs (= Hausdorff locally convex space) topology such that each inclusion $Z_n \subseteq Z$ is continuous, and we write

$$Z = \varinjlim_{n\in\mathbb{N}} Z_n$$

Notes:

- (LB)-spaces are closed under taking quotients with closed subspace, but generally not under taking closed subspaces.
- Not every dual of a Fréchet space is an (LB)-spaces.
- Surjective mappings between (LB)-spaces are open.

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Examples

- Let $a = (a_j)_{j\in I} \in \mathbb{R}_+^I$. We consider the Banach spaces, for $q \in [1,\infty]$,

$$\ell^p(a) = \{c = (c_j)_{j\in I} \in \mathbb{C}^I \mid (a_j c_j)_{j\in I} \in \ell^p(I)\}$$

If $A = (a_n)_{n\in\mathbb{N}} \in (\mathbb{R}_+^I)^{\mathbb{N}}$ is such that

$$a_{n+1,j} \leq a_{n,j}, \qquad \forall n \in \mathbb{N}, j \in I$$

then we define the (LB)-space, for $q \in [1,\infty]$,

$$k^p(A) = \varinjlim_{n\in\mathbb{N}} \ell^p(a_n)$$

- The space $C_c(\Omega)$ of all compactly supported continuous functions on a locally compact space.
- The space $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions.

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(Weak) acyclicity

Let $Z = \varinjlim_{n\in\mathbb{N}} Z_n$ be an (LB)-space. We may consider its injective resolution:

$$0 \longrightarrow \bigoplus_{n\in\mathbb{N}} Z_n \xrightarrow{d} \bigoplus_{n\in\mathbb{N}} Z_n \xrightarrow{\sigma} Z \longrightarrow 0$$

with

$$d((z_n)_{n\in\mathbb{N}}) = (z_n - z_{n-1})_{n\in\mathbb{N}}, \qquad \sigma((z_n)_{n\in\mathbb{N}}) = \sum_{n\in\mathbb{N}} z_n$$

Z is called (weakly) acyclic if d is (weakly) open.

Proposition (Palamodov, 1971; Debrouwere, N., 2024)

An (LB)-space Z is (weakly) acyclic if and only if every short exact sequence

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of (LB)-spaces is (weakly) topologically exact.

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A characterization for splitting

Theorem (Debrouwere, N., 2024)

The following are equivalent:

I. Z is weakly acyclic and every short exact sequence

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of (LB)-spaces splits.

II. Z is weakly E-acyclic, i.e., for every short exact sequence

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of (LB)-spaces, the map

$$j_* : L(Y, E) \to L(X, E), \quad T \mapsto T \circ j$$

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Weak E-acyclicity

Assumptions on E

From now on, we will assume E is a lcHs such that:

- it has a fundamental sequence $\{B_N : N \in \mathbb{N}\}$ of bounded sets, i.e., for every bounded set $B \subset E$ there exists a $N \in \mathbb{N}$ for which $B \subseteq B_N$.
- it is *locally complete*, i.e., for every bounded set $B \subset E$ there exists a Banach disk containing B.

An (LB)-space always has a fundamental sequence of bounded sets, but is in general **not** locally complete.

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Weak E-acyclicity vs. (weak) acyclicity

Let I denote an index set.

Theorem

Z is weakly $\ell^\infty(I)$-acyclic if and only if

- Z is weakly acyclic, if $|I| < \infty$;
- Z is acyclic, if $|I| \geq |Z'|$,

if and only if Z is weakly $\ell^\infty(I')$-acyclic for any index set I'.

Corollary

- Z is weakly acyclic if and only if Z is weakly $\mathbb{C}$-acyclic.
- If Z is weakly E-acyclic, then, Z is weakly acyclic.

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Decomposition of short exact sequences

For any (LB)-space $Z = \varinjlim_{n\in\mathbb{N}} Z_n$ and lcHs E we have that

$$L(Z,E) = \bigcap_{n\in\mathbb{N}} L(Z_n,E).$$

Lemma

Suppose we have a short exact sequence of (LB)-space

$$0 \longrightarrow X \xrightarrow{j} Y \xrightarrow{q} Z \longrightarrow 0$$

Then, $X = \varinjlim_{n\in\mathbb{N}} X_n$, $Y = \varinjlim_{n\in\mathbb{N}} Y_n$, and $Z = \varinjlim_{n\in\mathbb{N}} Z_n$ such that we have the short exact sequences of Banach spaces

$$0 \longrightarrow X_n \xrightarrow{j_{|X_n}} Y_n \xrightarrow{q_{|Y_n}} Z_n \longrightarrow 0 \ , \qquad n \in \mathbb{N}.$$

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Factorization through ℓ^1 and ℓ^∞

Suppose that either

$$\forall n \in \mathbb{N} \;\; \exists m \geq n : \qquad E_n \subseteq E_m \text{ factors through } \ell^\infty(I)$$

or

$$\forall n \in \mathbb{N} \; \exists m \geq n : \qquad Z_n \subseteq Z_m \text{ factors through } \ell^1(I)$$

Then

$$\forall n \in \mathbb{N} \; \exists m \geq n : \qquad L(X_m, E)_{|X_n} \subseteq (j_n)_*(L(Y_n, E))$$

This is the case when

- $E \cong k^\infty(A)$ or is nuclear;
- $Z \cong k^1(A)$ or is nuclear.

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Projective spectra

Let $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ be a sequence of vector spaces such that $U_{n+1} \subseteq U_n$.
Consider the mapping

$$\Psi_{\mathcal{U}} : \prod_{n\in\mathbb{N}} U_n \to \prod_{n\in\mathbb{N}} U_n, \quad (u_n)_{n\in\mathbb{N}} \to (u_{n+1} - u_n)_{n\in\mathbb{N}}.$$

We define the projective limit of $\mathcal{U}$ as

$$U = \operatorname{Proj} \mathcal{U} = \ker \Psi_{\mathcal{U}}.$$

We also consider the first derived projective limit functor of $\mathcal{U}$:

$$\operatorname{Proj}^1 \mathcal{U} = \left(\prod_{n\in\mathbb{N}} U_n\right) / \operatorname{Im} \Psi_{\mathcal{U}}.$$

$L(Z, E)$ is a projective limit

If $Z = \varinjlim_{n\in\mathbb{N}} Z_n$, then,

$$L(Z, E) \cong \operatorname{Proj}(L(Z_n, E))_{n\in\mathbb{N}}$$

as vector spaces.

Projective spectra

Let $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ be a sequence of vector spaces such that $U_{n+1} \subseteq U_n$. Consider the mapping

$$\Psi_{\mathcal{U}} : \prod_{n\in\mathbb{N}} U_n \to \prod_{n\in\mathbb{N}} U_n, \quad (u_n)_{n\in\mathbb{N}} \to (u_{n+1} - u_n)_{n\in\mathbb{N}}.$$

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The Mittag-Leffler procedure

Let $\mathcal{W} = (W_n)_{n\in\mathbb{N}}$ and $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ be projective spectra.

A sequence of linear mappings $f = (f_n)_{n\in\mathbb{N}}$, $f_n : W_n \to V_n$, is called a morphism between $\mathcal{W}$ and $\mathcal{V}$ if $(f_n)_{|W_m} = f_m$ for $m \geq n$. We then find the linear map

$$\operatorname{Proj} f : \operatorname{Proj} \mathcal{W} \to \operatorname{Proj} \mathcal{V}, \quad (w_n)_{n\in\mathbb{N}} \mapsto (f_n(w_n))_{n\in\mathbb{N}}.$$

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Let $f : \mathcal{W} \to \mathcal{V}$ be a morphism such that

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The vanishing of Proj^1 vs. weak E-acyclicity

Theorem (Debrouwere, N., 2024)

Suppose one of the following holds:

$$(i)\ E \cong k^\infty(A); \quad (ii)\ E \text{ is nuclear}; \quad (iii)\ Z \cong k^1(A); \quad (iv)\ Z \text{ is nuclear}.$$

Then

$$Z \text{ is weakly } E\text{-acyclic} \iff \mathrm{Proj}^1(L(Z_n, E))_{n\in\mathbb{N}} = 0.$$

The vanishing of Proj^1 : necessary and sufficient conditions

Let $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ be a projective spectrum with $U_n = \varinjlim_{N\in\mathbb{N}} U_{n,N}$.
Let $B_{n,N}$ be the unit ball in $U_{n,N}$.

Theorem (Vogt, 1989; Frerick, Wengenroth, 1996; Langenbruch, 2002)

$$\forall n \in \mathbb{N}\ \exists m \geq n\ \forall k \geq m\ \exists N \in \mathbb{N}\ \forall M \geq N, \varepsilon > 0\ \exists K \geq M, C > 0$$
$$B_{m,M} \subseteq C \cdot B_{k,K} + \varepsilon \cdot B_{n,N}$$

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The conditions (S) and (WS)

Suppose B_N is the unit ball of E_N.

- (E, Z) satisfies (S) if

$$\forall n \in \mathbb{N}\ \exists m \geq n\ \forall k \geq m\ \exists N \in \mathbb{N}\ \forall M \geq N, \varepsilon > 0\ \exists K \geq M, C > 0$$
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$$(E, Z) \text{ satisfies } (S) \iff (E, Z) \text{ satisfies } (WS) \text{ and } Z \text{ is acyclic}$$

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The characterization of weak E-acyclicity

Theorem (Debrouwere, N., 2024)

Suppose one of the following holds:

(i) $E \cong k^\infty(A)$; (ii) E is nuclear; (iii) $Z \cong k^1(A)$; (iv) Z is nuclear.

The following are equivalent:

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If E is an (LB)-space:

- Z is weakly acyclic, and every exact sequence

$$0 \longrightarrow E \xrightarrow{\iota} W \xrightarrow{Q} Z \longrightarrow 0$$

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The conditions (A) and $(D\Omega)$

- E satisfies (A) if

$$\exists N \in \mathbb{N}\ \forall M \geq N\ \forall \nu > 0\ \exists K \geq M, C > 0\ \forall r > 0 :$$
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- Z satisfies $(D\Omega)$ if

$$\forall n \in \mathbb{N}\ \exists m \geq n\ \forall k \geq m\ \exists \theta \in (0,1), C > 0 :$$
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If E satisfies (A) and Z satisfies $(D\Omega)$, then, (E, Z) satisfies (S).

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The problem of parameter dependence

The problem of parameter dependence

Suppose we have a continuous linear surjective map

$$Q : G \to H$$

Given data $(y_\lambda)_\lambda \subset H$ depending *regularly* on the parameter λ, do there exist corresponding solutions $x_\lambda \in G$ such that

$$Q(x_\lambda) = y_\lambda, \qquad \forall \lambda$$

depending on λ with the same regularity?

Example (Examples of regularity)

- Smoothly: $\Omega \ni \lambda \to y_\lambda \in H$ is a smooth function ($\Omega \subseteq \mathbb{R}^d$ open)
- Holomorphically: $\Omega \ni \lambda \to y_\lambda \in H$ is a holomorphic function ($\Omega \subseteq \mathbb{C}^d$ open)
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Vector-valued spaces: The ε-product

The problem is thus equivalent to solving a "*vector-valued version of* Q".

We put

$$W\varepsilon E = L(W'_c, E)$$

endowed with the strong topology.

Note that

- $W\varepsilon\mathbb{C} \cong W$;
- $W\varepsilon E \cong E\varepsilon W$ by transposition;
- When W and E are complete and one of them is nuclear we have

$$W\varepsilon E \cong W\widehat{\otimes}_\pi E \cong W\widehat{\otimes}_\varepsilon E$$

- When W is Montel, then, $W\varepsilon E = L(W'_b, F)$.

If W_1, W_2 and E_1, E_2 are lcHs, $A \in L(W_1, W_2)$, and $B \in L(E_1, E_2)$, then we define

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- When W is Montel, then, $W\varepsilon E = L(W'_b, F)$.

If W_1, W_2 and E_1, E_2 are lcHs, $A \in L(W_1, W_2)$, and $B \in L(E_1, E_2)$, then we define

$$A\varepsilon B : W_1\varepsilon E_1 \to W_2\varepsilon E_2, \quad L \mapsto B \circ L \circ A^t.$$

Vector-valued spaces: The ε-product

The problem is thus equivalent to solving a "*vector-valued version of* Q".

We put

$$W\varepsilon E = L(W'_c, E)$$

endowed with the strong topology.

Note that

- $W\varepsilon\mathbb{C} \cong W$;
- $W\varepsilon E \cong E\varepsilon W$ by transposition;
- When W and E are complete and one of them is nuclear we have

$$W\varepsilon E \cong W\widehat{\otimes}_\pi E \cong W\widehat{\otimes}_\varepsilon E$$

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The problem of parameter dependence – reformulation

The problem of parameter dependence

For any short exact sequence

$$0 \longrightarrow F \xrightarrow{\iota} G \xrightarrow{Q} H \longrightarrow 0 \ ,$$

for which lcHs E is the operator

$$Q\varepsilon \operatorname{id}_E : G\varepsilon E \to H\varepsilon E$$

surjective?

We will consider this problem when F, G, and H are Fréchet spaces.

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We will consider this problem when F, G, and H are Fréchet spaces.

Let us review some cases:

I. E is Fréchet and F, G, H are nuclear

Fact: the π-tensor product of two surjective continuous linear maps between Fréchet spaces is again surjective.

$\hookrightarrow$ The problem of parameter dependence is always solvable in this case.

II. E'_b is a Fréchet space

Then $F \varepsilon E = L(E'_b, F)$. (Similar for G and H.)

$\hookrightarrow$ We can use Vogt's results on splitting for Fréchet spaces.

! In this context, it is unnatural to impose additional assumptions on E.

III. F, G, H are Schwartz spaces

Then, F'_b is an (LB)-space and $F \varepsilon E \cong L(F'_b, E)$. (Similar for G and H.)

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The problem of parameter dependence – solution

Theorem (Debrouwere, N., 2024)

Let

$$0 \longrightarrow F \xrightarrow{\iota} G \xrightarrow{Q} H \longrightarrow 0$$

be a short exact sequence of Fréchet-Schwartz spaces.

Suppose one of the following holds:

$$(i)\ E \cong k^\infty(A); \quad (ii)\ E \text{ is nuclear}; \quad (iii)\ F_b' \cong k^1(A); \quad (iv)\ F \text{ is nuclear}.$$

If (E, F_b') satisfies (S), then,

$$Q\varepsilon \operatorname{id}_E : G\varepsilon E \to H\varepsilon E$$

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Theorem (Vogt, 1978; Debrouwere, N., 2024)

Let

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If

- F is nuclear and satisfies (Ω)
- E satisfies (A)

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Application: Vector-valued Eidelheit families

Eidelheit sequences

Let G be a Fréchet space and $(x'_n)_{n\in\mathbb{N}} \subseteq G'$. Consider the infinite system of scalar-valued linear equations on G:

$$\langle x'_n, x\rangle = c_n, \qquad n \in \mathbb{N}. \tag{1}$$

Eidelheit (1936) characterized when for each $(c_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ there exists a $x \in G$ solving (1). In this case $(x'_n)_{n\in\mathbb{N}}$ is called an Eidelheit sequence on G.

We may consider the continuous linear mapping

$$Q : G \to \mathbb{C}^{\mathbb{N}}, \qquad Q(x) = (\langle x'_n, x\rangle)_{n\in\mathbb{N}}$$

Then $(x'_n)_{n\in\mathbb{N}}$ is an Eidelheit sequence if and only if Q is surjective.

Note: Q cannot have a continuous linear right inverse.

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Vector-valued Eidelheit sequences

For any sequence $(x'_n)_{n\in\mathbb{N}} \subseteq G'$ and lcHs E we may also consider the infinite system of E-valued linear equations on $G\varepsilon E$:

$$\mathbf{x}(x'_n) = e_n, \qquad n \in \mathbb{N}. \tag{2}$$

We call (x'_n) E-Eidelheit if for each $(e_n)_{n\in\mathbb{N}} \in E^{\mathbb{N}}$ there exists $\mathbf{x} \in G\varepsilon E$ solving (2).

Consider the operator

$$Q\varepsilon \operatorname{id}_E : G\varepsilon E \to \mathbb{C}^{\mathbb{N}}\varepsilon E \cong E^{\mathbb{N}}$$

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A characterization

A Fréchet space F satisfies ($\underline{DN}$) if

$$\exists n \in \mathbb{N}\ \forall m \geq n\ \exists k \geq m, \theta \in (0,1), C > 0\ \forall x \in F :$$
$$\|x\|_m \leq C\|x\|_k^{\theta}\|x\|_n^{1-\theta}$$

Theorem (Debrouwere, N., 2024)

Let G be a nuclear Fréchet space satisfying ($\underline{DN}$). Let $(x_n') \subset G'$ be an Eidelheit sequence such that

$$\{x \in G \mid \langle x_n', x\rangle = 0,\ \forall n \in \mathbb{N}\}$$

satisfies (Ω). The following are equivalent:

- E satisfies (A);
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A sufficient condition for (Ω)

A Fréchet algebra $(F, *)$ is called *m*-convex if there exists an increasing fundamental sequence of seminorms $(\|\cdot\|_n)_{n\in\mathbb{N}}$ on F such that

$$\|x * y\|_n \leq \|x\|_n \|y\|_n, \qquad \forall x, y \in F,\ n \in \mathbb{N}.$$

Theorem (Braun, 1987)

Let $(G, *)$ be an *m*-convex Fréchet algebra and $(x'_n) \subset G'$ be an Eidelheit sequence such that each $x'_n : (G, *) \to \mathbb{C}$ is an algebra homomorphism.

Then, if G satisfies (Ω), so does

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Application: E. Borel's theorem

Theorem (Debrouwere, N., 2024)

The following are equivalent:

- E satisfies (A).
- For each sequence $(e_\alpha)_{\alpha\in\mathbb{N}^d} \subset E$ there is $\mathbf{f} \in C^\infty(\mathbb{R}^d; E)$ such that

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Application: interpolation by holomorphic functions

Let $U \subseteq \mathbb{C}$ be open and define

$$\mathcal{O}(U;E) = \{f : U \to E \mid f \text{ is holomorphic } \}.$$

Theorem

Let $U \subseteq \mathbb{C}$ be open and connected. Let $(u_n)_{n\in\mathbb{N}} \subseteq U$ be a sequence without accumulation points in U and let $(m_n)_{n\in\mathbb{N}}$ be a sequence of natural numbers.

The following are equivalent:

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Application: the vector-valued Stieltjes moment problem

The Stieltjes moment problem

The Stieltjes moment problem asks whether one can find for any given sequence $(c_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ a non-decreasing function $\theta(t)$ on $[0,\infty)$ such that

$$\int_0^\infty t^n d\theta(t) = c_n, \qquad \forall n \in \mathbb{N}$$

- The problem was solved by Stieltjes in 1884.
- In 1938 Boas and Polya found that there always exists a function θ of *bounded variation*.
- In 1989 Durán showed that one may always choose $d\theta(t) = f(t)dt$ with $f \in \mathcal{S}(\mathbb{R}_+)$, $\mathbb{R}_+ = (0,\infty)$, where

$$\mathcal{S}(\mathbb{R}_+) = \{g \in C^\infty(\mathbb{R}_+) \mid \sup_{t>0} |t^k g^{(m)}(t)| < \infty,\ \forall k,m \in \mathbb{N}\}.$$

- In 1991 he improved the result by giving a characterization of the solvability of the Stieltjes moment problem for general exponents $\{z_n\}_{n\in\mathbb{N}}$.

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The Stieltjes moment problem

The Stieltjes moment problem asks whether one can find for any given sequence $(c_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ a non-decreasing function $\theta(t)$ on $[0,\infty)$ such that

$$\int_0^\infty t^n d\theta(t) = c_n, \qquad \forall n \in \mathbb{N}$$

- The problem was solved by Stieltjes in 1884.
- In 1938 Boas and Polya found that there always exists a function θ of *bounded variation*.
- In 1989 Durán showed that one may always choose $d\theta(t) = f(t)dt$ with $f \in \mathcal{S}(\mathbb{R}_+)$, $\mathbb{R}_+ = (0,\infty)$, where

$$\mathcal{S}(\mathbb{R}_+) = \{g \in C^\infty(\mathbb{R}_+) \mid \sup_{t>0} |t^k g^{(m)}(t)| < \infty,\ \forall k,m \in \mathbb{N}\}.$$

- In 1991 he improved the result by giving a characterization of the solvability of the Stieltjes moment problem for general exponents $\{z_n\}_{n\in\mathbb{N}}$.

Theorem (Debrouwere, N., 2024)

Let Λ be an index set and $w_j : \Lambda \to (0, \infty)$ such that $w_j(\lambda) \leq w_{j+1}(\lambda)$.
The following are equivalent:

(i) $\exists j \; \forall k \geq j \; \exists l \geq k \; \forall \lambda \in \Lambda \; : \; w_k(\lambda)^2 \lesssim w_j(\lambda) w_l(\lambda)$.

(ii) For every sequence $(c_{n,\lambda}) \in \mathbb{C}^{\mathbb{N} \times \Lambda}$ satisfying

$$\forall n \in \mathbb{N} \; \exists j \in \mathbb{N} \; : \; |c_{n,\lambda}| \lesssim w_j(\lambda) \tag{3}$$

there exist measurable functions $f_\lambda : \mathbb{R}_+ \to \mathbb{C}$, $\lambda \in \Lambda$, such that

$$\forall n \in \mathbb{N} \; \exists j \in \mathbb{N} \; : \; \int_0^\infty |t^n f_\lambda(t)| dt \lesssim w_j(\lambda)$$

and

$$\int_0^\infty t^n f_\lambda(t) dt = c_{n,\lambda}, \qquad \forall n \in \mathbb{N}, \lambda \in \Lambda. \tag{4}$$

(iii) We may take the solutions $f_\lambda \in \mathcal{S}(\mathbb{R}_+)$ such that

$$\forall k, m \in \mathbb{N} \; \exists j \in \mathbb{N} \; : \; \sup_{t>0} |t^k f_\lambda^{(m)}(t)| \lesssim w_j(\lambda)$$

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Example (Examples of weights)

Suppose $(\Lambda, \|\cdot\|)$ is a normed space and $\omega : [0, \infty) \to [0, \infty)$. E.g.,

* $\omega = \log(1 + |\cdot|)$;
* $\omega = |\cdot|^s$ for $s > 0$.

The weights $w_j(\lambda) = e^{j\omega(\|\lambda\|)}$ always do satisfy the condition (i) of the previous theorem.

The weights $w_j(\lambda) = e^{-\frac{1}{j}\omega(\|\lambda\|)}$ do not satisfy the condition (i) of the previous theorem.

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The Pettis integral

A function $\mathbf{f} : \mathbb{R}_+ \to E$ is called Pettis integrable if:

- For each $e' \in E$ we have that $\langle e', \mathbf{f} \rangle \in L^1(\mathbb{R}_+)$.
- There exists some $e \in E$ such that

$$\langle e', e \rangle = \int_0^\infty \langle e', \mathbf{f}(t) \rangle \, dt.$$

We then write

$$\int_0^\infty \mathbf{f}(t) dt = e.$$

The E-valued Stieltjes moment problem

We consider the E-valued Stieltjes moment problem:

Does there exists for every given sequence $(e_n)_{n\in\mathbb{N}} \in E^{\mathbb{N}}$ a function $\mathbf{f} : \mathbb{R}_+ \to E$ such that $t^n\mathbf{f}(t)$ is Pettis integrable for each $n \in \mathbb{N}$ and

$$\int_0^\infty t^n\mathbf{f}(t)dt = e_n, \qquad n \in \mathbb{N}?$$

Regularisation

Lemma

Suppose $\mathbf{f} : \mathbb{R}_+ \to E$ is such that $t^n \mathbf{f}(t)$ is Pettis integrable for each $n \in \mathbb{N}$.

For each $\psi \in \mathcal{S}(\mathbb{R}_+)$ we have that the Mellin convolution

$$\mathbf{f} *_M \psi(t) := \int_0^\infty \mathbf{f}(x)\psi\left(\frac{t}{x}\right)\frac{dx}{x}$$

exists and for every $e' \in E'$ we have $\langle e', f *_M \psi\rangle \in \mathcal{S}(\mathbb{R}_+)$.

Moreover

$$\int_0^\infty t^n \mathbf{f} *_M \psi(t)dt = \int_0^\infty t^n \psi(t)dt \cdot \int_0^\infty t^n \mathbf{f}(t)dt.$$

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The space $\mathcal{S}(\mathbb{R}_+; E)$

By $\mathcal{S}(\mathbb{R}_+; E)$ we mean the space of all $\mathbf{f} \in C^\infty(\mathbb{R}_+; E)$ such that for all $k, m \in \mathbb{N}$ and continuous seminorm p on E,

$$\sup_{t>0}(1+t)^k p(\mathbf{f}^{(m)}(t)) < \infty,$$

endowed with its natural lcHs topology.

Lemma

- A function $\mathbf{f} : \mathbb{R}_+ \to E$ belongs to $\mathcal{S}(\mathbb{R}_+; E)$ if and only if $\langle e', \mathbf{f}\rangle \in \mathcal{S}(\mathbb{R}_+)$ for any $e' \in E$.
- The mapping

$$\Phi : \mathcal{S}(\mathbb{R}_+)\varepsilon E \to \mathcal{S}(\mathbb{R}_+; E), \quad L \mapsto (t \mapsto L(\delta_t))$$

is an isomorphism.

$\Longrightarrow$ It suffices to solve the E-valued Stieltjes moment problem in $\mathcal{S}(\mathbb{R}_+; E)$.

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Vector-valued moments

Lemma

For any $n \in \mathbb{N}$ the mapping

$$\mathcal{M}_{E,n} : \mathcal{S}(\mathbb{R}_+; E) \to E, \quad \mathbf{f} \mapsto \int_0^\infty t^n \mathbf{f}(t) dt$$

is well-defined as a Pettis integral.

The following diagram commutes:

$$\begin{array}{ccc} \mathcal{S}(\mathbb{R}_+)\varepsilon E & \xrightarrow{\mathcal{M}_{\mathbb{C},n}\varepsilon\, \mathrm{id}_E} & \mathbb{C}^{\mathbb{N}}\varepsilon E \\ \Phi \| & & \| \\ \mathcal{S}(\mathbb{R}_+; E) & \xrightarrow{\mathcal{M}_{E,n}} & E^{\mathbb{N}} \end{array}$$

The space $\mathcal{S}(\mathbb{R}_+)$ – Topological properties

Proposition

$$\mathcal{S}(\mathbb{R}_+) \cong s\widehat{\otimes}\Lambda_0(n),$$

where

$$s = \{(c_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \mid \sup_{n\in\mathbb{N}} |c_n| n^k < \infty,\ \forall k \in \mathbb{N}\}$$

$$\Lambda_0(n) = \{(c_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \mid \sup_{n\in\mathbb{N}} |c_n| e^{-\gamma n} < \infty,\ \forall \gamma > 0\}$$

Corollary

$\mathcal{S}(\mathbb{R}_+)$ is nuclear and satisfies (Ω) and $(\underline{DN})$.

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$$f *_M g(t) = \int_0^\infty f(x) g\left(\frac{t}{x}\right) \frac{dx}{x}.$$

Proposition

$(\mathcal{S}(\mathbb{R}_+), *_M)$ is an m-convex Fréchet algebra. For any $f, g \in \mathcal{S}(\mathbb{R}_+)$ we have

$$\mathcal{M}_{\mathbb{C},n}(f *_M g) = \mathcal{M}_{\mathbb{C},n}(f) \cdot \mathcal{M}_{\mathbb{C},n}(g), \qquad n \in \mathbb{N}.$$

Corollary

The closed subspace

$$\{f \in \mathcal{S}(\mathbb{R}_+) \mid \mathcal{M}_{\mathbb{C},n}(f) = 0, \ \forall n \in \mathbb{N}\}$$

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Theorem (Debrouwere, N., 2024)

The following are equivalent:

(i) E satisfies (A).

(ii) The E-valued Stieltjes moment problem is solvable.

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Thank you for your attention!

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