The splitting of exact sequences of (LB)-spaces

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Harmonic analysis, spaces and operators
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Introduction

Let X, Y, Z be lcHs (= Hausdorff locally convex spaces), $\iota \in L(Y, X)$ and $Q \in L(X, Z)$.

We say that the complex

$$0 \longrightarrow Y \xrightarrow{\iota} X \xrightarrow{Q} Z \longrightarrow 0$$

is a short exact sequence if

- $\iota : Y \to X$ is a continuous linear injection
- $Q : X \to Z$ is a continuous linear surjection
- $\operatorname{Im} \iota = \ker Q$

We say that the complex is topologically exact if ι and Q are topological homomorphisms.

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Consider a topologically short exact sequence

$$0 \longrightarrow Y \xrightarrow{\iota} X \xrightarrow{Q} Z \longrightarrow 0$$

We say that the complex splits if one (all) of the following equivalent conditions is satisfied:

1. There exists a continuous linear map $R : Z \to X$ such that $Q \circ R = \mathrm{id}_Z$
2. There exists a continuous linear map $L : X \to Y$ such that $L \circ \iota = \mathrm{id}_Y$
3. Y is isomorphic to a complemented subspace of X
4. Z is isomorphic to a complemented subspace of X

Intuitively, it says that for the solvable problem

$$Q(x) = z$$

one can "choose the solutions in a continuous way".

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Some history:

- In the 1960s, Palamodov applied the theory of the derived functor from homological algebra to find concrete necessary and sufficient conditions for splitting.
- In the 1970s and 80s, building upon the work of Palamodov, Vogt and Wagner provided a fairly complete characterization of splitting in the category of Fréchet spaces.
- In recent times, primarily by the work of Bonet and Domański, characterizations for splitting in the category of (PLS)-spaces have been found in specific cases.
- Until now, no satisfactory analogue of the results of Vogt and Wagner were found for the category of (LB)-spaces. This is because the category is **not quasi-abelian** (it is not closed under taking closed subspaces), hence one cannot apply the theory of the derived functor directly.

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Splitting in the category of (LB)-spaces and weak E-acyclicity

Let $(Z_n)_{n\in\mathbb{N}}$ be Banach spaces and Z a vector space such that

$$Z_n \subseteq Z_m \subseteq Z, \qquad \forall n \leq m$$

and

$$Z = \bigcup_{n\in\mathbb{N}} Z_n$$

We say that Z is an (LB)-space if Z is endowed with the finest (Hausdorff) locally convex topology such that each inclusion $Z_n \subseteq Z$ is continuous, and we write

$$Z = \varinjlim_{n\in\mathbb{N}} Z_n$$

Lemma

An (LB)-space Z is nuclear if and only if

$$Z \cong F'_\beta$$

for some nuclear Fréchet space F.

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Let $a = (a_j)_{j\in I} \in \mathbb{R}_+^I$. We consider the Banach spaces, for $q \in [1,\infty]$,

$$\ell^p(a) = \{c = (c_j)_{j\in I} \in \mathbb{C}^I \mid (a_j c_j)_{j\in I} \in \ell^p(I)\}$$

and

$$c_0(a) = \{c = (c_j)_{j\in I} \in \mathbb{C}^I \mid (a_j c_j)_{j\in I} \in c_0(I)\}$$

If $A = (a_n)_{n\in\mathbb{N}} \in (\mathbb{R}_+^I)^{\mathbb{N}}$ is such that

$$a_{n+1,j} \leq a_{n,j}, \qquad \forall n \in \mathbb{N}, j \in I$$

then we define the (LB)-space, for $q \in [1,\infty]$,

$$k^p(A) = \varinjlim_{n\in\mathbb{N}} \ell^p(a_n)$$

Lemma

$k^p(A)$ is nuclear if and only if

$$\forall I' \subseteq I \text{ countable } \forall n \in \mathbb{N}\ \exists m \geq n : \quad \sum_{j\in I'} \frac{a_{m,j}}{a_{n,j}} < \infty$$

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Let $Z = \varinjlim_{n\in\mathbb{N}} Z_n$ be an (LB)-space. We may consider its injective resolution:

$$0 \longrightarrow \bigoplus_{n\in\mathbb{N}} Z_n \xrightarrow{d} \bigoplus_{n\in\mathbb{N}} Z_n \xrightarrow{\sigma} Z \longrightarrow 0$$

with

$$d((z_n)_{n\in\mathbb{N}}) = (z_n - z_{n-1})_{n\in\mathbb{N}}, \qquad \sigma((z_n)_{n\in\mathbb{N}}) = \sum_{n\in\mathbb{N}} z_n$$

Z is called (weakly) acyclic if d is (weakly) open.

Proposition (Palamodov)

An (LB)-space Z is (weakly) acyclic if and only if every short exact sequence

$$0 \longrightarrow Y \xrightarrow{\iota} X \xrightarrow{Q} Z \longrightarrow 0$$

of (LB)-spaces is (weakly) topologically exact.

Lemma

Z is nuclear $\Rightarrow$ Z is the dual of a quasinormable Fréchet space $\Rightarrow$ Z is acyclic

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Z is nuclear $\Rightarrow$ Z is the dual of a quasinormable Fréchet space $\Rightarrow$ Z is acyclic

Theorem

Let E be an (LB)-space and Z an acyclic (LB)-space. The following are equivalent:

I. Every short exact sequence

$$0 \longrightarrow E \longrightarrow X \longrightarrow Z \longrightarrow 0$$

of (LB)-spaces splits.

II. Z is weakly E-acyclic, i.e., for every short exact sequence

$$0 \longrightarrow Y \xrightarrow{\iota} X \xrightarrow{Q} Z \longrightarrow 0$$

of (LB)-spaces, the complex

$$0 \longrightarrow L(Z,E) \xrightarrow{\circ Q} L(X,E) \xrightarrow{\circ \iota} L(Y,E) \longrightarrow 0$$

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Theorem (Debrouwere, N.)

Suppose that

- E is a non-normable, sequentially complete lcHs with countable fundamental system of bounded sets $\{B_N\}_{N\in\mathbb{N}}$
- $Z \cong \varinjlim_{n\in\mathbb{N}} Z_n$ is an (LB)-space

Assume that one of the following conditions is satisfied:

$$(i)\ E \cong k^\infty(A), \quad (ii)\ E \text{ is nuclear}, \quad (iii)\ Z \cong k^1(A), \quad (iv)\ Z \text{ is nuclear}$$

Then, the following are equivalent:

A. Z is weakly E-acyclic.

B. (E, Z) satisfies (WS), i.e.,

$$\forall n \in \mathbb{N}\ \exists m \geq n\ \forall k \geq m\ \exists N \in \mathbb{N}\ \forall M \geq N\ \exists K \geq M, S > 0$$
$$\forall z \in Z_n : \quad \|z\|_{Z_m} B_M \subseteq S\left(\|z\|_{Z_k} B_K + \|z\|_{Z_n} B_N\right).$$

Remark

If E is non-normable and (E, Z) satisfies (WS), then, Z is acyclic.

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$$\forall n \in \mathbb{N}\ \exists m \geq n\ \forall k \geq m\ \exists N \in \mathbb{N}\ \forall M \geq N\ \exists K \geq M, S > 0$$
$$\forall z \in Z_n : \quad \|z\|_{Z_m} B_M \subseteq S\left(\|z\|_{Z_k} B_K + \|z\|_{Z_n} B_N\right).$$

Remark

If E is non-normable and (E, Z) satisfies (WS), then, Z is acyclic.

In the case E is a Banach space the result still holds if:

- $Z \cong k^1(A)$, or,
- Z is nuclear

If $E \cong \ell^\infty(I)$ for some index set I, the result depends on the cardinality of I:

Theorem

Z is weakly $\ell^\infty(I)$-acyclic if and only if

- Z is weakly acyclic, if $|I| < \infty$
- Z is acyclic, if $|I| \geq |Z'|$,

if and only if Z is weakly $\ell^\infty(I')$-acyclic for any index set I'

Remark

If $|I| = \kappa$, κ a cardinal number, then Z being weakly $\ell^\infty(I)$-acyclic is equivalent to Z being so-called "κ-regular".

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Let $(U_n)_{n\in\mathbb{N}}$ be vector spaces such that $U_{n+1} \subseteq U_n$. Consider the mapping

$$\Psi : \prod_{n\in\mathbb{N}} U_n \to \prod_{n\in\mathbb{N}} U_n, \quad (u_n)_{n\in\mathbb{N}} \mapsto (u_{n+1} - u_n)_{n\in\mathbb{N}}$$

We define its (algebraic) projective limit as

$$U = \varprojlim_{n\in\mathbb{N}} U_n = \ker \Psi$$

and put

$$\operatorname{Proj}^1 U = \left(\prod_{n\in\mathbb{N}} U_n\right) / \operatorname{Im} \Psi$$

The Abstract Mittag-Leffler Procedure

Let $f : W \to V$ be a morphism between two projective limits $W = \varprojlim_{n\in\mathbb{N}} W_n$ and $V = \varprojlim_{n\in\mathbb{N}} V_n$ such that

$$\forall n \in \mathbb{N}\ \exists m \geq n : \quad V_m \subseteq f(W_n)$$

If U is the kernel of f, then, if $\operatorname{Proj}^1 U = 0$, the mapping f is surjective.

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In the case where U is a (PLB)-space, i.e.,

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each $U_{n,N}$ a Banach space, we find the following sufficient and necessary conditions.

Theorem

$$\forall n \in \mathbb{N}\ \exists m \geq n\ \forall k \geq m\ \exists N \in \mathbb{N}\ \forall M \geq N, \varepsilon > 0\ \exists K \geq M, C > 0$$
$$U_{m,M} \subseteq C \cdot U_{k,K} + \varepsilon \cdot U_{n,N}$$

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We say that a Banach space F is called weakly E-acyclic in the category of Banach spaces if for each short exact sequence

$$0 \longrightarrow H \xrightarrow{\iota} G \xrightarrow{Q} F \longrightarrow 0$$

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Lemma

A Banach space F is weakly E-acyclic in the category of Banach spaces if:

- F is a projective Banach space (e.g. $l^1(I)$)
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An (LB)-space $Z = \varinjlim_{n\in\mathbb{N}} Z_n$ is locally weakly E-acyclic if each Z_n is weakly E-acyclic in the category of Banach spaces.
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$$L(Z,E) \cong \varprojlim_{n\in\mathbb{N}} L(Z_n,E)$$

as vector spaces.

Theorem

If Z is a locally weakly E-acyclic, then,

$$Z \text{ is weakly } E\text{-acyclic} \iff \operatorname{Proj}^1 L(Z,E) = 0$$

Then, if E has a countable fundamental system of Banach disks $\{B_N\}_{N\in\mathbb{N}}$, we may view $L(Z,E)$ as a (PLB)-space:

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A lcHs E with fundamental system of bounded sets $\{B_N\}_{N\in\mathbb{N}}$ satisfies (A) if

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Theorem

If E satisfies (A) and Z satisfies (DΩ), then, (E, Z) satisfies (WS).

Remark

If either E or Z is isomorphic to the dual of a nuclear shift-invariant power series space (e.g. s'), then, the corresponding property for the other space is also necessary.

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Corollary (Debrouwere, N.)

Let E and Z be (LB)-spaces and suppose the following:

- E satisfies (A)
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Then, every short exact sequence

$$0 \longrightarrow E \longrightarrow X \longrightarrow Z \longrightarrow 0$$

of (LB)-spaces splits.

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The extension of short exact sequences of Fréchet-Montel spaces to ε-products

For a lcHs F we have

$$F \cong L_\beta(F'_c, \mathbb{C})$$

where F'_c denotes the dual of F endowed with the topology of uniform convergence on absolutely convex sets.

For any other lcHs E we may then define an "E-valued version of F" as follows:

$$F\varepsilon E = L_\beta(F'_c, E)$$

If

- F is Montel, then, $F\varepsilon E = L_\beta(F'_\beta, E)$
- either E or F is nuclear, then, $F\varepsilon E \cong F\widehat{\otimes}E$

Let F_0, F_1 and E_0, E_1 be lcHs and $A \in L(F_0, F_1)$ and $B \in L(E_0, E_1)$. We then define the continuous linear operator:

$$A\varepsilon B : F_0\varepsilon E_0 \to F_1\varepsilon E_1, \quad \psi \mapsto B \circ \psi \circ A^t$$

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$$A\varepsilon B : F_0\varepsilon E_0 \to F_1\varepsilon E_1, \quad \psi \mapsto B \circ \psi \circ A^t$$

For a lcHs F we have

$$F \cong L_\beta(F'_c, \mathbb{C})$$

where F'_c denotes the dual of F endowed with the topology of uniform convergence on absolutely convex sets.

For any other lcHs E we may then define an "E-valued version of F" as follows:

$$F\varepsilon E = L_\beta(F'_c, E)$$

If

- F is Montel, then, $F\varepsilon E = L_\beta(F'_\beta, E)$
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Theorem (Debrouwere, N.)

Suppose

$$0 \longrightarrow F \xrightarrow{\iota} G \xrightarrow{Q} H \longrightarrow 0$$

is a short exact sequence of Fréchet-Montel spaces and E is a sequentially complete lcHs with countable fundamental system of bounded sets.

Assume that one of the following conditions is satisfied:

$$(i)\ E \cong k^{\infty}(A), \quad (ii)\ E \text{ is nuclear}, \quad (iii)\ F \cong c_0(A), \quad (iv)\ F \text{ is nuclear}$$

If (E, F') satisfies (WS), then, the complex

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Application: the extension of the Stieltjes moment problem with complex exponents

The Stieltjes moment problem asks whether one can find for any given sequence $(a_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ a non-decreasing function $\theta(t)$ on $[0,\infty)$ such that

$$\int_0^\infty t^n d\theta(t) = a_n, \qquad \forall n \in \mathbb{N}$$

- This was solved by Stieltjes in 1884.
- In 1938 Boas and Polya found that there always exists a function θ of *bounded variation*.
- In 1989 Duran showed that one may always choose θ in the Schwartz space on the real half-line.
- In 1991 he improved the result by giving a characterization of the solvability of the Stieltjes moment problem for general exponents $\{z_n\}_{n\in\mathbb{N}}$.

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Take $-\infty \leq \beta < \alpha \leq +\infty$ and let E be any lcHs.

We define the space $S(\alpha, \beta; E)$ as all those $\varphi \in C^\infty((0,\infty); E)$ such that for any $p \in \mathrm{csn}(E)$, $k \in \mathbb{N}$, and $\beta + k + 1 < B < \alpha + k + 1$ we have

$$\sup_{t \in (0,\infty)} t^B p(\varphi^{(k)}(t)) < \infty$$

If $E = \mathbb{C}$, we simply write $\mathcal{S}(\alpha, \beta) = S(\alpha, \beta; \mathbb{C})$.

In the case where $\alpha = +\infty$ and $\beta = 0$ we find the the classical Schwartz space on the real half-line, i.e.,

$$S(\infty, 0) = \{\varphi \in \mathcal{S}(\mathbb{R}) \mid \varphi(t) = 0 \text{ if } t < 0\}$$

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If E is a sequentially complete lcHs, then,

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Then, we say that $(z_n)_{n\in\mathbb{N}}$ satisfies (S) if $z_n \neq z_m$ when $n \neq m$ and if either one of the following conditions holds:

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Theorem (Duran, 1991)

The following are equivalent:

(a) The sequence $(z_n)_{n\in\mathbb{N}}$ satisfies (S)

(b) For each complex sequence $(a_n)_{n\in\mathbb{N}}$ there exists a function φ such that $t^{z_n}\varphi(t)$ are Lebesgue integrable and $\int_0^\infty t^{z_n}\varphi(t) = a_n$ for all $n \in \mathbb{N}$

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Theorem (Debrouwere, N.)

Let E be a sequentially complete lcHs with countable fundamental system of bounded sets and $(z_n)_{n\in\mathbb{N}} \subset \mathbb{C}^{\mathbb{N}}$.

The following are equivalent:

(I) E satisfies (A) and $(z_n)_{n\in\mathbb{N}}$ satisfies (S)

(II) For every sequence $(e_n)_{n\in\mathbb{N}} \in E^{\mathbb{N}}$ there exists a function $f : \mathbb{R}_+ \to E$ such that $p \circ t^{z_n} f(t) \in L^1$ for every csn(E) and

$$e_n = \int_0^\infty t^{z_n} f(t)dt, \qquad \forall n \in \mathbb{N}$$

as a Pettis integral.

(III) For every sequence $(e_n)_{n\in\mathbb{N}} \in E^{\mathbb{N}}$ there exists a function $f \in \mathcal{S}(\alpha, \beta; E)$ such that

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