Structural theorems for the asymptotic behavior of ultradistributions

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Introduction (1)

Let $\mathcal{F}$ be a locally convex spaces of smooth functions on $\mathbb{R}^d$.
If the translation operator is continuous on $\mathcal{F}$, then for any $f \in \mathcal{F}'$ we define

$$\langle f(x+h), \varphi(x) \rangle = \langle f(x), \varphi(x-h) \rangle, \qquad \forall \varphi \in \mathcal{F}, h \in \mathbb{R}^d$$

If the dilation operator is continuous on $\mathcal{F}$, then for any $f \in \mathcal{F}'$ we define

$$\langle f(\lambda x), \varphi(x) \rangle = \frac{1}{\lambda^d} \langle f(x), \varphi(x/\lambda) \rangle, \qquad \forall \varphi \in \mathcal{F}, \lambda \in \mathbb{R}_+$$

General question

Suppose $f(x+h)$ (resp. $f(\lambda x)$) behaves in a certain way as $|h|$ (resp. λ) nears a limit point. What structure does this impose on f?

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Introduction (2)

- The idea of looking at translates of distributions goes back to Schwartz
- Pilipović and Stanković later introduced the so-called S-asymptotic behavior and thoroughly investigated its properties for distributions, ultradistributions, and Fourier hyperfunctions
- Quasi-asymptotic behavior (dilation) was first introduced for distributions by Zav'yalov (1973) and further developed by him, Drozhzhinov, and Vladimirov
- Complete structure theorems for 1-dimensional distributional quasi-asymptotics at infinity were found by Vindas (2008), and at the origin by Vindas and Pilipović (2009)
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Introduction (3)

Our results establish ultradistributional (first) structural theorems for several types of asymptotic behavior, either novel or under considerably weaker conditions then previously known.

This talk is based on the following articles:

A. Debrouwere, L. Neyt and J. Vindas, *On the space of ultradistributions vanishing at infinity*, submitted, arXiv: 1907.04205.

L. Neyt and J. Vindas, *Structural theorems for quasiasymptotics of ultradistributions*, Asymptot. Anal. 114 (2019), 118.

L. Neyt and J. Vindas, *A multidimensional Tauberian theorem for Laplace transforms of ultradistributions*, submitted, arXiv: 1902.00902.

L. Neyt and J. Vindas, *Asymptotic boundedness and moment asymptotic expansion in ultradistribution spaces*, submitted, arXiv: 1906.06232.

Weight sequences

- Let $M = (M_p)_{p\in\mathbb{N}}$ be a sequence of positive real numbers. We shall always assume that M satisfies

$(M.1)$ $M_p^2 \leq M_{p-1}M_{p+1}$, $p \geq 1$

$(M.2)'$ $M_{p+1} \leq AH^p M_p$, $p \in \mathbb{N}$, for some $A, H > 0$

In some cases we also assume the condition

$(M.3)'$ $\sum_{p=0}^{\infty} M_p/M_{p+1} < \infty$

- The associated function of M is defined as

$$M(t) := \sup_{p\in\mathbb{N}} \log \frac{t^p M_0}{M_p}, \quad t > 0, \quad M(0) := 0$$

- Example: $M_p = p!^\sigma$, $\sigma > 0$; then $M(t) \asymp t^{1/\sigma}$
  If $\sigma > 1$ this weight sequence satisfies $(M.3)'$
- $M \subset N$ means that $M_p \leq CL^p N_p$, $p \in \mathbb{N}$, for some $C, L > 0$

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Gelfand-Shilov spaces

Let M and A be two weight sequences. Set

$$\mathcal{S}^{M,\ell}_{A,q}(\mathbb{R}^d) := \{\varphi \in C^\infty(\mathbb{R}^d) : \sup_{\alpha\in\mathbb{N}^d}\sup_{x\in\mathbb{R}^d} \frac{|\varphi^{(\alpha)}(x)|e^{A(q|x|)}}{\ell^{|\alpha|}M_{|\alpha|}} < \infty\}$$

We then define

$$\mathcal{S}^{(M)}_{(A)}(\mathbb{R}^d) = \varprojlim_{\ell\to 0^+} \mathcal{S}^{M,\ell}_{A,1/\ell}(\mathbb{R}^d), \qquad \mathcal{S}^{\{M\}}_{\{A\}}(\mathbb{R}^d) = \varinjlim_{\ell\to\infty} \mathcal{S}^{M,\ell}_{A,1/\ell}(\mathbb{R}^d)$$

The duals $\mathcal{S}'^{(M)}_{(A)}(\mathbb{R}^d)$ and $\mathcal{S}'^{\{M\}}_{\{A\}}(\mathbb{R}^d)$ are called *tempered ultradistributions of Beurling and Roumieu type*, respectively. We denote $[\cdot]$ to consider $(\cdot)$ and $\{\cdot\}$ simultaneously, e.g. $\mathcal{S}'^{[M]}_{[A]}(\mathbb{R}^d)$.

$\mathcal{D}^{[M]}(\mathbb{R}^d)$ denotes the subset of compactly supported elements in $\mathcal{S}^{[M]}_{[A]}(\mathbb{R}^d)$, which is non-trivial and dense if M satisfies $(M.3)'$. Elements in its dual $\mathcal{D}'^{[M]}(\mathbb{R}^d)$ are called ultradistributions.

Assumption

We always assume $\mathcal{S}^{(M)}_{(A)}(\mathbb{R}^d)$ is non-trivial. A sufficient condition is, e.g, that $p!^\sigma \subset M$ and $p!^\tau \subset A$ for $\sigma + \tau > 1$

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S-asymptotics

Introduction (1)

Let $\omega : \mathbb{R}^d \to \mathbb{R}_+$ be continuous and Γ be a cone.
An ultradistribution $f \in \mathcal{D}'^{[M]}(\mathbb{R}^d)$ is said to have S-asymptotic behavior with respect to ω on Γ, with limit $g \in \mathcal{D}'^{[M]}(\mathbb{R}^d)$, if

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As this imposes a certain structure on g, we may consider the following general question.

General Question

Let $\mathcal{F}$ be a space of translation invariant smooth functions. Can we characterize, for a certain weight function $\omega : \mathbb{R}^d \to \mathbb{R}_+$, when for some $f \in \mathcal{F}'$

$$f * \varphi(h) = O(\omega(h)), \qquad \text{as } |h| \to \infty, \quad \forall\varphi \in \mathcal{F} \tag{1}$$

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- In his book *Théorie des distributions* L. Schwartz considered the case $\mathcal{F} = \mathcal{D}(\mathbb{R}^d)$, $\omega = 1$. He showed that for any $f \in \mathcal{D}'(\mathbb{R}^d)$:
  1. f satisfies (1) iff $f \in \mathcal{B}'(\mathbb{R}^d) := (\mathcal{D}_{L^1}(\mathbb{R}^d))'$
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- Dimovski, Pilipović, Prangoski and Vindas provided the ultradistributional second structure theorem by use of the parametrix method
- We obtained the ultradistributional first structure theorem under the conditions $(M.1)$ and $(M.2)'$ by use of the short-time Fourier transform

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  2. f satisfies (2) iff $f \in \dot{\mathcal{B}}'(\mathbb{R}^d) := \overline{\mathcal{D}(\mathbb{R}^d)}^{\mathcal{B}'(\mathbb{R}^d)}$
- Dimovski, Pilipović, Prangoski and Vindas provided the ultradistributional second structure theorem by use of the parametrix method
- We obtained the ultradistributional first structure theorem under the conditions $(M.1)$ and $(M.2)'$ by use of the short-time Fourier transform

short-time Fourier transform (1)

Take some $\psi \in \mathcal{S}_{(A)}^{(M)}(\mathbb{R}^d) \setminus \{0\}$. For any $f \in \mathcal{S}_{[A]}^{\prime[M]}(\mathbb{R}^d)$ we define the short-time Fourier transform (STFT) $V_\psi f$ as

$$V_\psi f(x,\xi) = \left\langle f, e^{-2\pi i \xi \cdot} \overline{\psi(\cdot - x)} \right\rangle = \int_{\mathbb{R}^d} f(t) \overline{\psi(t-x)} e^{-2\pi i \xi \cdot t} dt$$

For a weight sequence N we denote by $\mathcal{S}_{(N)}(\mathbb{R}^d)$ the Fréchet space of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that

$$\sup_{|\alpha| \le k} \sup_{x \in \mathbb{R}^d} |\varphi^{(\alpha)}(x)| e^{N(k|x|)} < \infty, \qquad \forall k \in \mathbb{N}.$$

Proposition

The following mappings are continuous

$$V_\psi : \mathcal{S}_{(A)}^{(M)}(\mathbb{R}^d) \to \mathcal{S}_{(A)}(\mathbb{R}^d_x) \widehat{\otimes} \mathcal{S}_{(M)}(\mathbb{R}^d_\xi)$$

and

$$V_\psi : \mathcal{S}_{(A)}^{\prime(M)}(\mathbb{R}^d) \to \mathcal{S}'_{(A)}(\mathbb{R}^d_x) \widehat{\otimes} \mathcal{S}'_{(M)}(\mathbb{R}^d_\xi)$$

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short-time Fourier transform (2)

We define the adjoint STFT of $F \in \mathcal{S}'_{(A)}(\mathbb{R}^d_x)\widehat{\otimes}\mathcal{S}'_{(M)}(\mathbb{R}^d_\xi)$ as

$$\left\langle V_\psi^* F, \varphi \right\rangle := \left\langle F, \overline{V_{\overline{\psi}}\varphi} \right\rangle, \qquad \varphi \in \mathcal{S}^{(M)}_{(A)}(\mathbb{R}^d)$$

Proposition

Let $\gamma \in \mathcal{S}^{(M)}_{(A)}(\mathbb{R}^d)$ be a synthesis window for ψ, i.e. $(\gamma, \psi)_{L^2} \neq 0$, then the reconstruction formula

$$\frac{1}{(\gamma,\psi)_{L^2}} V_\gamma^* \circ V_\psi = \mathrm{id}_{\mathcal{S}'^{(M)}_{(A)}(\mathbb{R}^d)}$$

is valid and the desingularization formula

$$\langle f, \varphi \rangle = \frac{1}{(\gamma,\psi)_{L^2}} \int\int_{\mathbb{R}^{2d}} V_\psi f(x,\xi) V_{\overline{\gamma}}\varphi(x,-\xi)\,dx\,d\xi$$

holds for all $f \in \mathcal{S}'^{[M]}_{[A]}(\mathbb{R}^d)$ and $\varphi \in \mathcal{S}^{[M]}_{[A]}(\mathbb{R}^d)$

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The space $\mathcal{D}^{[M]}_{L^1_\omega}$

Take continuous $\omega : \mathbb{R}^d \to \mathbb{R}_+$ which is (A)-admissible (resp. $\{A\}$-admissible), i.e.

$$\exists \lambda > 0 \text{ (resp. } \forall \lambda > 0) \ \exists C > 0 \ \forall x, t \in \mathbb{R}^d : \ \omega(x+t) \leq C\omega(x)e^{A(\lambda|t|)}$$

For $\ell > 0$ we denote by $\mathcal{D}^{M,\ell}_{L^1_\omega}$ the Banach spaces of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that

$$\sup_{\alpha \in \mathbb{N}^d} \frac{\int_{\mathbb{R}^d} |\varphi^{(\alpha)}(x)|\omega(x)dx}{\ell^{|\alpha|} M_{|\alpha|}} < \infty$$

We define

$$\mathcal{D}^{(M)}_{L^1_\omega} = \varprojlim_{\ell \to 0^+} \mathcal{D}^{M,\ell}_{L^1_\omega}, \qquad \mathcal{D}^{\{M\}}_{L^1_\omega} = \varinjlim_{\ell \to \infty} \mathcal{D}^{M,\ell}_{L^1_\omega}$$

Lemma

We have the following dense continuous inclusion

$$\mathcal{S}^{[M]}_{[A]}(\mathbb{R}^d) \hookrightarrow \mathcal{D}^{[M]}_{L^1_\omega} \hookrightarrow \mathcal{S}'^{[M]}_{[A]}(\mathbb{R}^d)$$

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The space $\mathcal{B}'^{[M]}_{\omega}$ (1)

We denote the dual of $\mathcal{D}^{[M]}_{L^1_\omega}$ by $\mathcal{B}'^{[M]}_{\omega}$

Theorem 1 (N. and Vindas, 2019)

Let $\psi \in \mathcal{S}^{(M)}_{(A)}(\mathbb{R}^d) \setminus \{0\}$. For any $f \in \mathcal{S}'^{[M]}_{[A]}(\mathbb{R}^d)$ TFAE:

(1) $f \in \mathcal{B}'^{[M]}_{\omega}$

(2) $f(x+h) = O(\omega(h))$, $h \in \mathbb{R}^d$, in $\mathcal{S}'^{[M]}_{[A]}(\mathbb{R}^d)$

(3) For some $q > 0$ (for all $q > 0$) it holds that

$$\sup_{(x,\xi)\in\mathbb{R}^{2d}} e^{-M(q|\xi|)} \frac{|V_\psi f(x,\xi)|}{\omega(x)} < \infty$$

If $\psi \in \mathcal{D}^{(M)}(\mathbb{R}^d) \setminus \{0\}$ and $f \in \mathcal{D}'^{[M]}(\mathbb{R}^d)$ then we have the extra equivalency

(4) $f(x+h) = O(\omega(h))$, $h \in \mathbb{R}^d$, in $\mathcal{D}'^{[M]}(\mathbb{R}^d)$

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The space $\mathcal{B}_\omega'^{[M]}$ (2)

Theorem 2 (N. and Vindas, 2019)

$f \in \mathcal{B}_\omega'^{[M]}$ if and only if there are continuous $\{f_\alpha\}_{\alpha\in\mathbb{N}^d}$ s.t. for some $\ell > 0$ (for any $\ell > 0$) there exists $C_\ell > 0$ s.t.

$$|f_\alpha(x)| \leq C_\ell \frac{\ell^{|\alpha|}}{M_\alpha}\omega(x), \qquad x \in \mathbb{R}^d, \alpha \in \mathbb{N}^d,$$

and

$$f = \sum_{\alpha\in\mathbb{N}^d} f_\alpha^{(\alpha)}$$

This follows from the Hahn-Banach theorem, where in the Roumieu case we employed the projective description

$$\mathcal{D}_{L^1_\omega}^{\{M\}} \cong \varprojlim_{(r_p)\in\mathfrak{R}} D_{L^1_\omega}^{M_p \prod_{j=1}^p r_j, 1}$$

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$$\mathfrak{R} := \{(r_p)_{p\in\mathbb{Z}_+} : r_p \nearrow \infty \text{ and } r_p > 0, \forall p \in \mathbb{Z}_+\}$$

The space $\mathcal{B}'^{[M]}_{\omega}$ (2)

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The space $\dot{\mathcal{B}}'^{[M]}_{\omega}$ (1)

We define $\dot{\mathcal{B}}'^{[M]}_{\omega}$ as the closure of $C_c(\mathbb{R}^d)$ in $\mathcal{B}'^{[M]}_{\omega}$. Equivalently, it is the closure of $\mathcal{S}^{[M]}_{[A]}(\mathbb{R}^d)$ in $\mathcal{B}'^{[M]}_{\omega}$.

Theorem 3 (Debrouwere, N. and Vindas, 2019)

Let $\psi \in \mathcal{S}^{(M)}_{(A)}(\mathbb{R}^d) \setminus \{0\}$. For any $f \in \mathcal{S}'^{[M]}_{[A]}(\mathbb{R}^d)$ TFAE:

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(4) $f(x+h) = o(\omega(h))$, $|h| \to \infty$, in $\mathcal{D}'^{[M]}(\mathbb{R}^d)$

The space $\dot{\mathcal{B}}'^{[M]}_{\omega}$ (2)

Theorem 4 (Debrouwere, N. and Vindas, 2019)

$f \in \dot{\mathcal{B}}'^{[M]}_{\omega}$ if and only if there are continuous $\{f_\alpha\}_{\alpha\in\mathbb{N}^d}$ s.t. for some $\ell > 0$ (for any $\ell > 0$) there exists $C_\ell > 0$ s.t.

$$|f_\alpha(x)| \leq C_\ell \frac{\ell^{|\alpha|}}{M_\alpha}\omega(x), \qquad x \in \mathbb{R}^d, \alpha \in \mathbb{N}^d,$$

the limits

$$\lim_{|x|\to\infty} \frac{f_\alpha(x)}{\omega(x)} = 0, \qquad \forall \alpha \in \mathbb{N}^d,$$

hold, and

$$f = \sum_{\alpha\in\mathbb{N}^d} f_\alpha^{(\alpha)}$$

Proof of Theorem 4 (1)

Let E be a Banach space. We denote by $\Lambda_{(M)}(E)$ (by $\Lambda_{\{M\}}(E)$) the space of all $(e_\alpha)_{\alpha\in\mathbb{N}^d} \in E^{\mathbb{N}^d}$ s.t. for some $\ell > 0$ (any $\ell > 0$)

$$\sup_{\alpha\in\mathbb{N}^d} \ell^{|\alpha|} M_\alpha \|e_\alpha\|_E < \infty$$

Claim

The following map is surjective

$$S : \Lambda_{[M]}(C_{0,\omega}(\mathbb{R}^d)) \to \dot{\mathcal{B}}'^{[M]}_\omega : \quad (f_\alpha)_{\alpha\in\mathbb{N}^d} \mapsto \sum_{\alpha\in\mathbb{N}^d} f_\alpha^{(\alpha)}$$

Lemma

Let $S : E \to F$ be a continuous linear map. Suppose E and $\operatorname{Im} S$ are Mackey and $E/\ker S$ is complete. Then S is surjective if

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Proof of Theorem 4 (2)

What is the dual $\dot{\mathcal{B}}'^{[M]}_{\omega}$? The space $\mathcal{D}^{[M]}_{L^1_\omega}$ is not semi-reflexive, i.e. $(\mathcal{B}'^{[M]}_{\omega})' \neq \mathcal{D}^{[M]}_{L^1_\omega}$.

Proposition

Let $\psi \in \mathcal{S}^{(M)}_{(A)}(\mathbb{R}^d) \setminus \{0\}$ and $f \in \mathcal{S}'^{[M]}_{[A]}(\mathbb{R}^d)$. Then $f \in \mathcal{D}^{[M]}_{L^1_\omega}$ iff

$$\forall q > 0 \text{ (resp. } \exists q > 0) : \quad \sup_{\xi \in \mathbb{R}^d} e^{M(q|\xi|)} \int_{\mathbb{R}^d} |V_\psi f(x,\xi)| \omega(x) dx < \infty$$

Theorem 5 (Debrouwere, N. and Vindas, 2019)

The canonical inclusion mapping

$$\iota : \mathcal{D}^{[M]}_{L^1_\omega} \to (\dot{\mathcal{B}}'^{[M]}_{\omega})'_b : \quad \varphi \mapsto (f \mapsto \langle f, \varphi \rangle)$$

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Proof of Theorem 4 (3)

- $E = \Lambda_{[M]}(C_{0,\omega}(\mathbb{R}^d))$ is a complete (LB)-space (a Fréchet space) so that E is Mackey and $E/\ker S$ is complete. $\operatorname{Im} S$ is infrabarreled (is Fréchet), whence Mackey
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$\Lambda'_{(M)}(E) := \Lambda_{\{1/M\}}(E)$, $\Lambda'_{\{M\}}(E) := \Lambda_{(1/M)}(E)$ and $\mathcal{M}^1_\omega = (C_{0,\omega}(\mathbb{R}^d))'$

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Structure of S-asymptotics

Let Γ be a convex cone and $\omega : \mathbb{R}^d \to \mathbb{R}_+$ continuous such that

$$\lim_{h\in\Gamma,|h|\to\infty} \omega(x+h)/\omega(x) \qquad \text{exists for all } x \in \mathbb{R}^d.$$

Theorem 6 (Debrouwere, N. and Vindas, 2019)

$f \in \mathcal{D}'^{[M]}(\mathbb{R}^d)$ has S-asymptotic behavior on Γ w.r.t. ω if and only if for each $R > 0$ there exists $f_\alpha \in C(\mathbb{R}^d)$ s.t.

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Quasi-asymptotics

Introduction

Let $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous.
An ultradistribution $f \in \mathcal{D}'^{[M]}(\mathbb{R})$ is said to have quasi-asymptotic behavior with respect to ρ, with limit $g \in \mathcal{D}'^{[M]}(\mathbb{R})$, if

$$\lim_{\lambda \to \infty} \frac{\langle f(\lambda x), \varphi(x) \rangle}{\rho(\lambda)} = \langle g(x), \varphi(x) \rangle , \qquad \varphi \in \mathcal{D}^{[M]}(\mathbb{R})$$

This imposes the following structure:

- $\rho(\lambda) = \lambda^a L(\lambda)$ for some slowly varying function L
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Some observations

Lemma

1. If $a > -1$ then quasi-asymptotic behavior is local : if f has quasi-asymptotics then so does $f - f_c$ for any $f_c \in \mathcal{E}'^{[M]}(\mathbb{R})$
2. If $a \notin \mathbb{Z}_-$: if f has quasi-asymptotics of degree a then for any Nth primitive F_N of f there is a polynomial p such that $F_N - p$ has quasi-asymptotics of degree $a + N$
3. If $a = -k$, $k \geq 2$: if f has quasi-asymptotics of degree $-k$ then there is a $(k-1)$th primtivie F that has quasi-asymptotics of degree -1

Conclusion: Structure Theorems for quasiasymptotics of degree ≥ -1 provide Structure Theorems for arbitrary degree!

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Case $a > -1$

- Suppose f vanishes near the origin $\Rightarrow g(x) := f(e^x)$ has S-asymptotics
- Use previous Structure Theorem on g
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Theorem 7 (N. and Vindas, 2018)

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2. One may similarly consider quasi-asymptotic boundedness. Analogous theorems hold.
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4. The structure of quasi-asymptotic behavior of tempered ultradistributions $\mathcal{S}'^{[M]}_{[A]}(\mathbb{R}^d)$ is not known. We have shown Tauberian theorems for the Laplace transform in [3].

Remarks

1. Similar structural theorems can be found for quasi-asymptotic behavior at the origin.
2. One may similarly consider quasi-asymptotic boundedness. Analogous theorems hold.
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The Moment Asymptotic Expansion

Introduction

Let $\mathcal{F}$ be a l.c.s. of smooth functions closed under dilation. An element $f \in \mathcal{F}'$ is said to satisfy the moment asymptotic expansion (MAE) in $\mathcal{F}'$ if there are $\mu_\alpha \in \mathbb{C}$ s.t. for any $\varphi \in \mathcal{F}$

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Using our previous results, we find:

Theorem 9 (N. and Vindas, 2019)

1. Any $f \in \mathcal{K}'^{[M]}(\mathbb{R}^d)$ satisfies the MAE in $\mathcal{K}'^{[M]}(\mathbb{R}^d)$.
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Uniform Moment Asymptotic Expansion (1)

An element $f \in \mathcal{F}'$ is said to satisfy the uniform moment asymptotic expansion (UMAE) in $\mathcal{F}'$ w.r.t. $[A]$ if there are $\mu_\alpha \in \mathbb{C}$ s.t. for any $\varphi \in \mathcal{F}$

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uniformly in $k \in \mathbb{N}$.

For weight sequence M, N, consider the space $\mathcal{K}^{[M]}_{[N]}(\mathbb{R}^d)$ of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that

$$\exists q > 0 \; \forall \ell > 0 \; (\exists \ell > 0 \; \forall q > 0): \quad \sup_{\alpha \in \mathbb{N}^d, x \in \mathbb{R}^d} \frac{e^{-N(q|x|)}(1+|x|)^{|\alpha|}|\varphi^{(\alpha)}(x)|}{\ell^{|\alpha|} M_{|\alpha|}} < \infty$$

We have to impose $(M.2)$ on our weight sequences.

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Uniform Moment Asymptotic Expansion (2)

Theorem 10 (N. and Vindas, 2019)

Set $A_p = N_p \max_{j \leq p}(M_j/j!)$. Any element $f \in \mathcal{K}'^{[M]}_{[N]}(\mathbb{R}^d)$ satisfies the UMAE in $\mathcal{K}'^{[M]}_{[N]}(\mathbb{R}^d)$ with respect to $[A]$.

If M satisfies $(M.1)$, $(M.2)$ and $(M.3)$, we have in the 1-dimensional case:

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