The Structure of Ultradistributional Quasiasymptotics

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Definition

Let $\mathcal{F}$ be a space of smooth functions and $f \in \mathcal{F}'$. Then f has quasiasymptotic behavior $g \in \mathcal{F}'$ with respect to the gauge function ρ if

$$f(\lambda x) \sim \rho(\lambda) g(x)$$

holds in $\mathcal{F}'$, that is, for any $\varphi \in \mathcal{F}$ we have

$$\langle f(\lambda x), \varphi(x) \rangle = \langle f(x), \varphi(x/\lambda)/\lambda \rangle = \rho(\lambda) \langle g(x), \varphi(x) \rangle + o(\rho(\lambda))$$

as $\lambda \to \infty$ (resp. as $\lambda \to 0+$).

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Lemma

1. $\rho(\lambda) = \lambda^{\alpha} L(\lambda)$ for some slowly varying function L
2. g is a homogeneous generalized function of degree α

Let M_p be some non-decreasing sequence of positive numbers. Consider the functions $\varphi \in C^\infty(\mathbb{R})$ such that for any compact $K \Subset \mathbb{R}$ and any $h > 0$ (resp. for some $h > 0$)

$$\sup_{n\in\mathbb{N}} \sup_{x\in K} \frac{\left|\varphi^{(n)}(x)\right|}{h^n M_n} < \infty$$

The space of all these functions will be denoted by $\mathcal{E}^{(M_p)}$ (resp. by $\mathcal{E}^{\{M_p\}}$). The subspace of all functions that have compact support will be denoted by $\mathcal{D}^{(M_p)}$ (resp. by $\mathcal{D}^{\{M_p\}}$).
We use $*$ as a common notation for (M_p) and $\{M_p\}$.

Elements in $\mathcal{D}^{*\prime}$ are called ultradistributions
The elements in $\mathcal{E}^{*\prime}$ are exactly those ultradistributions with compact support

Proposition

Let $f \in \mathcal{D}^{*\prime}$ be homogeneous of degree α.

1. If $\alpha \notin \mathbb{Z}_-$
$$f(x) = c_+ x_+^\alpha + c_- x_-^\alpha$$
2. If $\alpha \in \mathbb{Z}_-$
$$f(x) = c_1 x^\alpha + c_2 \delta^{(\alpha-1)}(x)$$

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Proposition - Moment Asymptotic Expansion

For any $f \in \mathcal{E}^{*\prime}$ there exist $\mu_n \in \mathbb{C}$, $n \in \mathbb{N}$, such that

$$f(\lambda x) \sim \sum_{n=0}^{\infty} \frac{(-1)^n \mu_n \delta^{(n)}(x)}{n! \lambda^{n+1}}$$

as $\lambda \to \infty$ in $\mathcal{E}^{*\prime}$