The Uniform Moment Asymptotic Expansion

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2. The Ultradistributional case
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1 Motivation: The case of Schwartz distributions

2 The Ultradistributional case

3 The Uniform Moment Asymptotic Expansion

4 Characterization of the UMAE

Consider a distribution $f \in \mathcal{D}'(\mathbb{R})$ and a test function φ. By the Taylor expansion

$$\varphi(x) = \sum_{n=0}^{N-1} \frac{\varphi^{(n)}(0)}{n!} x^n + \mathcal{O}\left(x^N\right),$$

as $x \to 0$.

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as $x \to 0$.

- One would 'expect' that as $\lambda \to \infty$

$$\langle f(\lambda x), \varphi(x) \rangle = \sum_{n=0}^{N-1} \frac{\varphi^{(n)}(0)}{n! \lambda^{n+1}} \langle f(x), x^n \rangle + \mathcal{O}\left(\lambda^{-(N+1)}\right)$$

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- **In general**, this does not hold. For example, for $\alpha \notin \mathbb{Z}$:

$$f(\lambda x) := (\lambda x - x_0)_+^\alpha \sim \sum_{k=0}^{\infty} (-1)^n \binom{\alpha}{n} x_0^n \frac{x_+^{\alpha-n}}{\lambda^{n-\alpha}},$$

as $\lambda \to \infty$.

Definition

A distribution $f \in \mathcal{D}'(\mathbb{R})$ is said to satisfy the **moment asymptotic expansion (MAE)**, or is **distributionally small at infinity**, if for certain $\mu_0, \mu_1, \mu_2, \ldots$ (called the **moments** of f), the following holds

$$f(\lambda x) \sim \sum_{n=0}^{\infty} \frac{(-1)^n \mu_n \delta^{(n)}(x)}{n! \lambda^{n+1}}, \quad \lambda \to \infty$$

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Examples

For each element of the following spaces the MAE holds:

1. $\mathcal{E}'$: distributions with compact support
2. $\mathcal{O}'_{\mathcal{C}}$: space of convolutors of $\mathcal{S}$
3. $\mathcal{K}'$

Definition

The space $\mathcal{K}_q$ is formed by those smooth functions ϕ s.t. $\phi^{(n)}(x) = \mathcal{O}\left(|x|^{q-n}\right)$ as $|x| \to \infty$ for each $n \in \mathbb{N}$. A topology is generated by the seminorms

$$\|\phi\|_{n,q} = \max\left\{ \sup_{|x|\leq 1} |\phi^n(x)|, \sup_{|x|\geq 1} |x^{n-q}\phi^{(n)}(x)| \right\}$$

Then

$$\mathcal{K} = \varinjlim \mathcal{K}_q$$

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$$\mathcal{K} = \varinjlim \mathcal{K}_q$$

Theorem (Estrada and Kanwal [2002])

Let $f \in \mathcal{D}'$. Then the following are equivalent:

1. f satisfies the MAE
2. $f \in \mathcal{K}'$

The MAE is an exceedingly useful behavior and may be applied in many cases such as:

- The simplification of proofs, for example:
  - Watson's lemma
  - Laplace's Asymptotic Formula
  - The Method of Steepest Descent

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- The regularization of distributions that do not satisfy the MAE, for instance the Stieltjes integral

$$\Phi(s) = \int_0^\infty \frac{\phi(x)dx}{x+s},$$

as $s \to 0+$ and $\phi \in \mathcal{S}$.

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- Obtaining new results in fields such as analytic combinatorics, number theory, transform analysis, ...

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4 Characterization of the UMAE

Let M_p be a sequence of positive numbers. We will always assume the conditions

(M.1) $M_p^2 \leq M_{p-1} M_{p+1}$ for each $p \in \mathbb{N}$

(M.3)' $\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty$

As usual, $*$ will stand for either the Beurling or Roumieu case.

Definition

Any $f \in \mathcal{D}^{*\prime}$ is said to satisfy the **Moment Asymptotic Expansion (MAE)**, or is **ultradistributionally small at infinity**, if there exist $\mu_0, \mu_1, \mu_2, \ldots$ such that for any $\varphi \in \mathcal{D}^*$ and $N \in \mathbb{N}$:

$$\langle f(\lambda x), \varphi(x) \rangle = \sum_{n=0}^{N-1} \frac{\mu_n \varphi^{(n)}(0)}{n! \lambda^{n+1}} + \mathcal{O}\left(\lambda^{-(N+1)}\right),$$

as $\lambda \to \infty$.

Example: The space $\mathcal{E}^{*\prime}$

For any $\varphi \in \mathcal{E}^*$ and $N \in \mathbb{N}$, consider the Nth Taylor approximation

$$\varphi_N(x) = \sum_{n=0}^{N-1} \frac{\varphi^{(n)}(0)}{n!} x^n.$$

Then for any $f \in \mathcal{E}^{*\prime}$, there exists some $R > 0$ and a $h_f > 0$ (any $h_f > 0$)

$$\begin{aligned}\langle f(\lambda x), \varphi(x) \rangle &= \langle f(\lambda x), \varphi_N(x) + (\varphi(x) - \varphi_N(x)) \rangle \\ &\leq \sum_{n=0}^{N-1} \frac{\mu_n \varphi^{(n)}(0)}{n! \lambda^{n+1}} + \frac{\|f\| \, \|(\varphi - \varphi_N)(x/\lambda)\|_{R,h_f}}{\lambda},\end{aligned}$$

with

$$\|(\varphi - \varphi_N)(x/\lambda)\|_{R,h_f} = \sup_{|x| \leq R, n \in \mathbb{N}} \frac{|D^n[\varphi - \varphi_N](x/\lambda)|}{h_f^n M_n}.$$

$$\|(\varphi - \varphi_N)(x/\lambda)\|_{R,h_f} \stackrel{?}{=} \mathcal{O}\left(\lambda^{-N}\right)$$

Via the properties of φ, we get for $n < N$ and any $h > 0$ (some $h > 0$):

$$\sup_{x \leq R} \left| D^n \left[\varphi\left(\frac{x}{\lambda}\right) - \varphi_N\left(\frac{x}{\lambda}\right) \right] \right| = \sup_{|x| \leq R} \frac{1}{\lambda^n} \left| \frac{\varphi^{(N)}(\xi_x/\lambda)}{(N-n)!} \frac{x^{N-n}}{\lambda^{N-n}} \right|$$

$$\leq \frac{C h^N M_N R^N}{\lambda^N} \leq \frac{C e^{M\left(\frac{1}{h_f}\right)} k^N M_N}{M_0} \frac{h_f^n M_n}{\lambda^N}$$

with $k = hR$. The same can also be concluded for $n \geq N$, hence there exists some $L > 0$ (independent of N) such that

$$\|(\varphi - \varphi_N)(x/\lambda)\|_{R,h_f} \leq L \left(\frac{k^N M_N}{M_0} \right) \frac{1}{\lambda^N}$$

for any $k > 0$ (for some $k > 0$).

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Definition

Any $f \in \mathcal{D}^{*\prime}$ is said to satisfy the **Uniform Moment Asymptotic Expansion (UMAE)** if there exist $\mu_0, \mu_1, \mu_2, \ldots$ such that for any $\varphi \in \mathcal{D}^*$, $N \in \mathbb{N}$ and $h > 0$ (for some $h > 0$),

$$\langle f(\lambda x), \varphi(x) \rangle = \sum_{n=0}^{N-1} \frac{\mu_n \varphi^{(n)}(0)}{n! \lambda^{n+1}} + \mathcal{O}\left(\frac{h^N M_N}{M_0} \lambda^{-(N+1)} \right)$$

holds uniformly.

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holds uniformly.

Theorem (Vindas, N.)

Any $f \in \mathcal{E}^{*\prime}$ satisfies the UMAE.

Observation

Suppose $f \in \mathcal{D}^{*\prime}$ satisfies the UMAE and is such that $\mu_n = 0$ for each $n \in \mathbb{N}$. Then for each $\varphi \in \mathcal{D}^*$ and any $h > 0$ (for some $h > 0$):

$$\langle f(\lambda x), \varphi(x) \rangle = \mathcal{O}\left(\frac{e^{-M(\lambda/h)}}{\lambda} \right) \quad , \text{ as } \lambda \to \infty$$

Proof. For any $\varphi \in \mathcal{D}^*$ there is a $C > 0$ such that for each $N \in \mathbb{N}$

$$| \langle f(\lambda x), \varphi(x) \rangle | \leq \frac{C}{\lambda} \left(\frac{h}{\lambda} \right)^N \frac{M_N}{M_0},$$

as $\lambda \to \infty$. Taking the infimum infers

$$\begin{aligned} | \langle f(\lambda x), \varphi(x) \rangle | &\leq \frac{C}{\lambda} \inf_{N \in \mathbb{N}} \left(\frac{h}{\lambda} \right)^N \frac{M_N}{M_0} = \frac{C}{\lambda} \left[\sup_{N \in \mathbb{N}} \left(\frac{\lambda}{h} \right)^N \frac{M_0}{M_N} \right]^{-1} \\ &= C \frac{e^{-M(\lambda/h)}}{\lambda} \end{aligned}$$

Corollary 1

For any periodic distribution f of zero mean, it holds that for each $\varphi \in \mathcal{D}^*$, $h > 0$ and $0 < \alpha < 1$ there is a $C > 0$ such that:

$$\langle f(\lambda x), \varphi(x) \rangle = \mathcal{O}\left(\frac{e^{-Ce^{\frac{1}{e\alpha}\left(\frac{\lambda}{h}\right)^{\alpha}}}}{\lambda} \right) \quad , \text{ as } \lambda \to \infty$$

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Corollary 2

For any two $f, g \in \mathcal{D}^{*\prime}$ that satisfy the UMAE and have the same moments, we have for any $h > 0$ (for some $h > 0$):

$$f(\lambda x) = g(\lambda x) + \mathcal{O}\left(\frac{e^{-M(\lambda/h)}}{\lambda} \right) \quad , \text{ as } \lambda \to \infty$$

In particular, there is always a $g \in \mathcal{S}$ such that f and g share equal moments.

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In the following we will always require that (M.2) and (M.3) hold as well as logarithmic convexity of the sequence $M_p^* = M_p/p!$.

Our goal is to show that under these conditions an ultradistribution f satisfies the MAE $\iff$ f satisfies the UMAE. This will be done according the following plan of action:

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Our goal is to show that under these conditions an ultradistribution f satisfies the MAE $\iff$ f satisfies the UMAE. This will be done according the following plan of action:

1. Construct a space $\mathcal{K}^{*\prime}$ of ultradistributions that all satisfy the UMAE.
2. Show that any $f \in \mathcal{D}^{*\prime}$ satisfying the MAE can be extended to an element of $\mathcal{K}^{*\prime}$.

The Space $\mathcal{K}^*$

Definition

For each $q \in \mathbb{N}$ and $h > 0$, we define the space $\mathcal{K}_q^{\{M_p\},h}$ as all smooth functions φ that allow the following norm

$$\|\varphi\|_{q,h} = \sup_{n\in\mathbb{N}} \max\left\{ \sup_{|x|\leq 1} \left| \frac{\varphi^{(n)}(x)}{h^n M_n} \right|, \sup_{|x|>1} \left| \frac{x^{n-q}\varphi^{(n)}(x)}{h^n M_n} \right| \right\}$$

Next, for each $q \in \mathbb{N}$ we define

$$\mathcal{K}_q^{(M_p)} = \varprojlim_{h\to 0} \mathcal{K}_q^{\{M_p\},h}, \qquad \mathcal{K}_q^{\{M_p\}} = \varinjlim_{h\to \infty} \mathcal{K}_q^{\{M_p\},h}$$

Finally

$$\mathcal{K}^* = \varinjlim \mathcal{K}_q^*$$

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Theorem (Vindas, N.)

Each $f \in \mathcal{K}^{*\prime}$ satisfies the UMAE.

MAE $\Rightarrow$ UMAE

Observation 1

We need only concern ourselves with those ultradistributions f for which $\operatorname{supp} f \subseteq (1, \infty)$. Consequently, it suffices to only consider test function $\varphi \in \mathcal{D}^*(0, \infty)$.

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Observation 2

Due to our conditions on the M_p, the following are true:

1. $g(x) = f(e^x)$ exists and $g \in D^{*\prime}$. [Eida and Pilipović (2005)]
2. For any $\psi \in \mathcal{D}^*$ there is a $\varphi \in \mathcal{D}^*(0, \infty)$ such that $\psi(x) = \varphi(e^x)$. [Hörmander (1990)]

$$\begin{aligned}\langle g(x+h), \psi(x)\rangle &= \langle f(e^x), \psi(x-h)\rangle = \left\langle f(e^x), \varphi\left(\frac{e^x}{e^h}\right)\right\rangle \\ &= \left\langle f(y), \varphi\left(\frac{y}{e^h}\right)\frac{e^h}{y}\frac{1}{e^h}\right\rangle = \left\langle f(e^h y), \frac{\varphi(y)}{y}\right\rangle \\ &= \mathcal{O}\left(e^{-Nh}\right),\end{aligned}$$

as $h \to \infty$ for any $N \in \mathbb{N}$.

$$\begin{aligned}\langle g(x+h), \psi(x)\rangle &= \langle f(e^x), \psi(x-h)\rangle = \left\langle f(e^x), \varphi\left(\frac{e^x}{e^h}\right)\right\rangle \\ &= \left\langle f(y), \varphi\left(\frac{y}{e^h}\right)\frac{e^h}{y}\frac{1}{e^h}\right\rangle = \left\langle f(e^h y), \frac{\varphi(y)}{y}\right\rangle \\ &= \mathcal{O}\left(e^{-Nh}\right),\end{aligned}$$

as $h \to \infty$ for any $N \in \mathbb{N}$.

In particular, for any $\psi \in \mathcal{D}^*$:

$$g * \psi(h) = \mathcal{O}\left(e^{-Nh}\right),$$

as $h \to \infty$ for any $N \in \mathbb{N}$.

Lemma (Vindas)

If for some $g \in \mathcal{D}^{*\prime}$ and some $N \in \mathbb{N}$

$$g * \psi(x) = \mathcal{O}\left(e^{-Nx}\right), \qquad \text{as } x \to \infty,$$

holds for any $\psi \in \mathcal{D}^*$, then there exist $g_k \in L^1$ such that

$$g = \sum_{k=0}^{\infty} (-1)^k g_k^{(k)},$$

and for some $h > 0$ (for any $h > 0$) there is a $C > 0$ such that

$$|g_k(x)| \leq C \frac{h^k e^{-Nx}}{M_k} \qquad , \forall k \in \mathbb{N}.$$

Furthermore, the g_k can be chosen such that for some $\alpha > 1$, $\operatorname{supp} g_k \subseteq [\alpha, \infty)$ for all $k \in \mathbb{N}$.

Take any $\varphi \in \mathcal{D}^*(0,\infty)$ and put $\nu(x) = e^x \varphi(e^x)$.

$$\langle f(y), \varphi(y) \rangle = \left\langle f(y), \frac{\varphi(y)}{y} y \right\rangle = \langle g(x), \nu(x) \rangle = \sum_{k=0}^{\infty} \int_{\alpha}^{\infty} g_k(x) \nu^{(k)}(x) dx$$

$$= \sum_{k=0}^{\infty} \int_{\alpha}^{\infty} g_k(x) \frac{d^k}{dx^k} [e^x \varphi(e^x)] dx$$

$$= \sum_{k=0}^{\infty} \int_{\alpha}^{\infty} g_k(x) \left(\sum_{j=0}^{k} \binom{k}{j} \sum_{m=0}^{j} \varphi^{(m)}(e^x) e^{mx} S(j,m) \right) dx$$

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Ultimately

$$\langle f(y), \varphi(y) \rangle = \sum_{m=0}^{\infty} \sum_{k=m}^{\infty} S(k+1, m+1) \int_{\log \alpha}^{\infty} g_k(\log y) \varphi^{(m)}(y) y^m dy$$

Take a $\varphi \in \mathcal{K}^*$. Then there is a $q \in \mathbb{N}$ such that for all $h_1 > 0$ (for some $h_1 > 0$) and $x \geq \log \alpha$

$$|\varphi^{(n)}(x)| \leq C h_1^n M_n x^{q-n} \qquad , \forall n \in \mathbb{N}$$

On the other hand, we have that for some $h_2 > 0$ (for any $h_2 > 0$)

$$|g_k(x)| \leq C' \frac{h_2^k e^{-(q+2)x}}{M_n} \qquad , \forall k \in \mathbb{N}$$

If we then "evaluate" f on φ, we get

$$\begin{aligned} &|\langle f(y), \varphi(x) \rangle| \\ &\leq CC' \sum_{m=0}^{\infty} \sum_{k=m}^{\infty} S(k+1, m+1) \int_{\log \alpha}^{\infty} \frac{h_2^k y^{-q-2}}{M_k} (h_1^m M_m y^{q-m}) y^m dy \\ &= \frac{CC'}{\log \alpha} \sum_{m=0}^{\infty} h_1^m M_m \sum_{k=m}^{\infty} S(k+1, m+1) \frac{h_2^k}{M_k} \end{aligned}$$

Lemma

If the M_p satisfy (M.1) and (M.3)', then for any $m \in \mathbb{N}$ and $h > 0$, there is a $k > 0$ and $C_h > 0$ (independent of m) such that

$$\sum_{k=m}^{\infty} S(k+1, m+1) \frac{h^k}{M_k} \leq C_h \frac{k^m}{M_m}$$

If the h can be chosen arbitrarily small, then so can the k.

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The previous lemma infers

$$|\langle f(y), \varphi(y) \rangle| \leq \frac{CC'C_k}{\log \alpha} \sum_{m=0}^{\infty} (h_1 k)^m,$$

and since either h_1 or k can be chosen freely, it follows that

$$|\langle f(y), \varphi(y) \rangle| \leq \frac{2CC'C_k}{\log \alpha}$$

In Conlusion

Theorem (Vindas, N.)

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Theorem (Vindas, N.)

Any $f \in \mathcal{D}^{*\prime}$ satisfies the UMAE if and only if there exist continuous functions f_k such that

$$f = \sum_{k=0}^{\infty} f_k^{(k)},$$

and

$$|f_k(x)| \leq C \frac{h_k}{M_k} x^k e^{-M(Ax)},$$

for some $C > 0, h > 0$ and any $A > 0$ (for any $h > 0$ and some $A > 0$).