Ultradistributional convolutor spaces

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Introduction

Let X be a space of smooth functions such that the convolution operation

$$* : X \times X \to X : \quad (\psi, \varphi) \mapsto \psi * \varphi(t) = \int_{\mathbb{R}^d} \varphi(x)\psi(t-x)dx$$

is well-defined and continuous.

Then, we can define the hypocontinuous convolution operator $* : X' \times X \to X'$ by transposition:

$$\langle f * \psi, \varphi \rangle := \langle f, \check{\psi} * \varphi \rangle, \qquad f \in X',\ \psi, \varphi \in X$$

Let E be a locally convex subspace of X'. For any $f \in X'$ one could ask: when does

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We define the space

$$\mathcal{O}'_C(X,E) := \{f \in X' : f * \varphi \in E, \forall \varphi \in X\}$$

If $f \in \mathcal{O}'_C(X,E)$ (and E is webbed and X' is ultrabornological), then it follows from the closed graph theorem that the linear map

$$*_f : X \to E : \quad \varphi \mapsto f * \varphi$$

is continuous. Hence, we may topologize $\mathcal{O}'_C(X,E)$ via the embedding

$$\mathcal{O}'_C(X,E) \to L_b(X,E) : \quad f \mapsto *_f$$

Main question

Can we characterize the structure and topology of $\mathcal{O}'_C(X,E)$?

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- We will consider this problem for ultradistributions. Moreover, we wish to include the quasianalytic case, so X will be the **Gelfand-Shilov space** and X' the **space of tempered ultradistributions**.

- Let $T_x f = f(\cdot - x)$ and $M_\xi f = e^{2\pi i \xi \cdot} f(\cdot)$ denote the translation and modulation operators respectively. As

$$T_x(f * \psi) = f * (T_{-x}\psi)$$

it follows that if X is closed is under translation, then so is

$$\{f * \varphi : f \in \mathcal{O}'_C(X, E), \varphi \in X\}$$

Hence we may assume E is translation invariant. In this talk we will consider the specific case where E is a so-called **translation-modulation invariant Banach space of ultradistributions**.

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Due to our low assumptions on the test function space, we can not use the **parametrix** method.

In our paper:

A. Debrouwere, L. Neyt, J. Vindas, *On the space of ultradistributions vanishing at infinity*, Banach J. Math. Anal., 14 (2020), 915–934,

we developed a novel technique using tools from **time-frequency analysis** to solve the question for the specific case $E = C_{0,\omega}(\mathbb{R}^d)$.

In this talk, we discuss the extension of this framework to the general case. The talk is based on the recent preprint:

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Preliminaries - Weight sequences

We call a sequence $M = (M_p)_{p\in\mathbb{N}} \in (\mathbb{R}_+)^{\mathbb{N}}$ a **weight sequence** if $\lim_{p\to\infty}(M_p/M_0)^{1/p} = \infty$. We define its associated function on $\mathbb{R}^d$ as

$$\omega_M(x) = \sup_{p\in\mathbb{N}} \log \frac{|x|^p M_0}{M_p}$$

Assumptions

We always assume the following two conditions for a weight sequence M:

$(M.1)$ $M_p^2 \leq M_{p-1}M_{p+1}$, $p \geq 1$;

$(M.2)'$ $M_{p+1} \leq C_0 H^p M_p$ for some $C_0, H \geq 1$.

In some cases we will strengthen $(M.2)'$ to:

$(M.2)$ $M_{p+q} \leq C_0 M_p M_q$ for some $C_0, H \geq 1$.

Classical examples are the Gevrey sequences $M_{G,s} = (p!^s)_{p\in\mathbb{N}}$ for any $s > 0$. They satisfy $(M.1)$ and $(M.2)$ and $\omega_{M_{G,s}}(x) \asymp |x|^{1/s}$.

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For two weight sequences M and A and $q, \ell > 0$ we define the Banach space

$$\mathcal{S}_{A,q}^{M,\ell} = \mathcal{S}_{A,q}^{M,\ell}(\mathbb{R}^d) := \left\{ \varphi \in C^\infty(\mathbb{R}^d) : \sup_{(\alpha,x)\in\mathbb{N}^d\times\mathbb{R}^d} \frac{|\varphi^{(\alpha)}(x)|e^{\omega_A(qx)}}{\ell^{|\alpha|}M_{|\alpha|}} < \infty \right\}$$

Then, we define the **Gelfand-Shilov spaces of Beurling type (of Roumieu type)** as

$$\mathcal{S}_{(A)}^{(M)} = \varprojlim_{\ell\to 0^+} \mathcal{S}_{A,\ell^{-1}}^{M,\ell}, \qquad \mathcal{S}_{\{A\}}^{\{M\}} = \varinjlim_{\ell\to\infty} \mathcal{S}_{A,\ell^{-1}}^{M,\ell}$$

and use $\mathcal{S}_{[A]}^{[M]}$ as a common notation (similar for other spaces and definitions).

Its strong dual, denoted by $\mathcal{S}_{[A]}'^{[M]}$, is also called the **space of tempered ultradistributions**.

The convolution mapping

$$* : \mathcal{S}_{[A]}^{[M]} \times \mathcal{S}_{[A]}^{[M]} \to \mathcal{S}_{[A]}^{[M]}$$

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TF Analysis – Vector-valued frequency spaces

Let X be a locally convex space and $w : \mathbb{R}^d \to (0, \infty)$ be continuous. We define $C_w(X)$ as the locally convex space of all continuous functions $\Phi : \mathbb{R}^d \to X$ such that

$$\sup_{\xi \in \mathbb{R}^d} w(\xi) p(\Phi(\xi)) < \infty, \qquad \forall p \in \operatorname{csn}(X)$$

Then, for any weight sequence M, we define the spaces

$$C_{(M)}(X) = \varprojlim_{q \to \infty} C_{e^{\omega_M(q\cdot)}}(X), \quad C_{\{M\}}(X) = \varprojlim_{q \to 0^+} C_{e^{\omega_M(q\cdot)}}(X),$$

and

$$C_{(M),d}(X) = \varinjlim_{q \to \infty} C_{e^{-\omega_M(q\cdot)}}(X), \quad C_{\{M\},d}(X) = \varinjlim_{q \to 0^+} C_{e^{-\omega_M(q\cdot)}}(X),$$

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TF Analysis – STFT

For any $f \in \mathcal{S}'^{[M]}_{[A]}$ and $\psi \in \mathcal{S}^{[M]}_{[A]}$ we define the **short-time Fourier transform (STFT)** as

$$V_\psi f(\xi) := (M_{-\xi} f) * \check{\overline{\psi}}$$

Proposition

If $\psi \in \mathcal{S}^{(M)}_{(A)}$, the linear map

$$V_\psi : \mathcal{S}'^{[M]}_{[A]} \to C_{[M],d}(\mathcal{S}'^{[M]}_{[A]})$$

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TF Analysis – Adjoint STFT

For any $\Phi \in C_{[M],d}(\mathcal{S}'^{[M]}_{[A]})$ and $\psi \in \mathcal{S}^{[M]}_{[A]}$ we define the **adjoint STFT** as the Pettis integral

$$V^*_\psi \Phi := \int_{\mathbb{R}^d} M_\xi [\Phi(\xi) * \psi] d\xi$$

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If $\psi = \psi_0 * \psi_1$ for certain $\psi_0, \psi_1 \in \mathcal{S}^{(M)}_{(A)}$, the linear map

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TF Analysis – Reconstruction

Proposition

For any $\psi, \gamma \in \mathcal{S}_{(A)}^{(M)}$ such that $(\gamma, \psi)_{L^2} \neq 0$, we have

$$\frac{1}{(\gamma, \psi)_{L^2}} V_\gamma^* \circ V_\psi = \mathrm{id}_{\mathcal{S}'^{[M]}_{[A]}}$$

and

$$\langle f, \varphi \rangle = \frac{1}{(\gamma, \psi)_{L^2}} \int_{\mathbb{R}^d} \langle V_\psi f(\xi), V_{\overline{\gamma}} \varphi(-\xi) \rangle \, d\xi, \qquad \forall f \in \mathcal{S}'^{[M]}_{[A]}, \varphi \in \mathcal{S}^{[M]}_{[A]}$$

TMIB – Definition

A Banach space E is called a **translation-modulation invariant Banach space of ultradistributions (TMIB)** of class $[M]$ - $[A]$ if it satisfies the following three conditions:

(I) the continuous dense inclusions $\mathcal{S}_{[A]}^{[M]} \hookrightarrow E \hookrightarrow \mathcal{S}_{[A]}'^{[M]}$ hold;

(II) $T_x(E) \subseteq E$ and $M_\xi(E) \subseteq E$ for all $x, \xi \in \mathbb{R}^d$;

(III) there exist $\tau, C > 0$ (for every $\tau > 0$ there exists a $C > 0$) such that

$$\omega_E(x) := \|T_x\|_{L(E)} \leq Ce^{\omega_A(\tau x)} \quad \text{and} \quad \nu_E(\xi) := \|M_{-\xi}\|_{L(E)} \leq Ce^{\omega_M(\tau \xi)}$$

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TMIB – Dual TMIB

If a Banach space $E = E_0'$ is the dual of TMIB E_0, then the space E is called a **dual translation-modulation invariant Banach space of ultradistributions (DTMIB)**. Note that a DTMIB E satisfies (II) and (III) and

$$\omega_E = \check{\omega}_{E_0} \quad \text{and} \quad \nu_E = \nu_{E_0}.$$

Moreover, we have that

$$\mathcal{S}_{[A]}^{[M]} \subset E \hookrightarrow \mathcal{S}_{[A]}'^{[M]},$$

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TMIB – Operations

Let E be a TMIB or DTMIB. We have the following continuous bilinear mappings

$$* : E \times L^1_{\omega_E} \to E : \quad (f, g) \mapsto f * g := \int_{\mathbb{R}^d} g(x) T_x f dx,$$

$$\cdot : E \times A_{\nu_E} \to E : \quad (f, h) \mapsto f \cdot h := \int_{\mathbb{R}^d} (\mathcal{F}^{-1} h)(\xi) M_{-\xi} f d\xi,$$

where $A_{\nu_E} := \mathcal{F}L^1_{\nu_E}$ and the integrals are interpreted as Pettis integrals.

TMIB – Examples

We fix a measurable function $\omega : \mathbb{R}^N \to (0, \infty)$ $(N = d, 2d)$ for which ω and ω^{-1} are locally bounded and so that

$$\omega(x + t) \leq C\omega(x)e^{\omega_A(qt)}, \qquad \exists q > 0 \ (\forall q > 0)$$

The following are examples of TMIB or DTMIB:

- The spaces L^p_ω, $L^{p,q}_\omega$, $p, q \in [1, \infty]$, and the space $C_{0,\omega}$ of continuous functions in L^∞_ω vanishing at infinity.
- Let F be a TMIB or DTMIB in $\mathbb{R}^{2d}$. The modulation space

$$\mathcal{M}^F = \{f \in \mathcal{S}'^{[M]}_{[A]} : V_\psi f \in F\}$$

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TMIB – Examples

We fix a measurable function $\omega : \mathbb{R}^N \to (0, \infty)$ ($N = d, 2d$) for which ω and ω^{-1} are locally bounded and so that

$$\omega(x + t) \leq C\omega(x)e^{\omega_A(qt)}, \qquad \exists q > 0 \ (\forall q > 0)$$

The following are examples of TMIB or DTMIB:

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TMIB – Associated spaces

We consider the following two spaces associated to a TMIB or DTMIB E:

- The ultradifferentiable space $\mathcal{D}_E^{[M]}$ is the space of all $f \in \mathcal{S}'^{[M]}_{[A]}$ such that $f^{(\alpha)} \in E$ for every $\alpha \in \mathbb{N}^d$ and

$$\sup_{\alpha \in \mathbb{N}^d} \frac{\|f^{(\alpha)}\|_E}{\ell^{|\alpha|} M_{|\alpha|}} < \infty, \qquad \forall \ell > 0 \ (\exists \ell > 0)$$

  endowed with its natural Fréchet space (resp. (LB)-space) topology.

- The convolutor space $\mathcal{O}'_C(\mathcal{S}^{[M]}_{[A]}, E)$ of all $f \in \mathcal{S}'^{[M]}_{[A]}$ such that

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TMIB – Assumptions

- M and A are weight sequences satisfying $(M.1)$ and $(M.2)'$ such that $\mathcal{S}_{(A)}^{(M)} \neq \{0\}$.
- E is a TMIB or DTMIB of class $[M] - [A]$ for which

$$\nu_E(\xi) e^{\omega_M(q_0\xi)} \leq C e^{\omega_M(q_1\xi)}, \qquad \forall q_0\ \exists q_1\ (\forall q_1\ \exists q_0)$$

  This is true if
  1. $\sup_{\xi\in\mathbb{R}^d} \nu_E(\xi)/(1+|\xi|)^k < \infty$ for some $k > 0$;
  2. M satisfies $(M.2)$.
- We always assume $\psi = \psi_0 * \psi_1$ and $\gamma = \gamma_0 * \gamma_1$ for certain $\psi_j, \gamma_j \in \mathcal{S}_{(A)}^{(M)}$.

  This condition is trivial if $\mathcal{S}_{(A)}^{(M)}$ has the strong factorization property, i.e. if $\mathcal{S}_{(A)}^{(M)} * \mathcal{S}_{(A)}^{(M)} = \mathcal{S}_{(A)}^{(M)}$. Conditions for this to be true can be found in:

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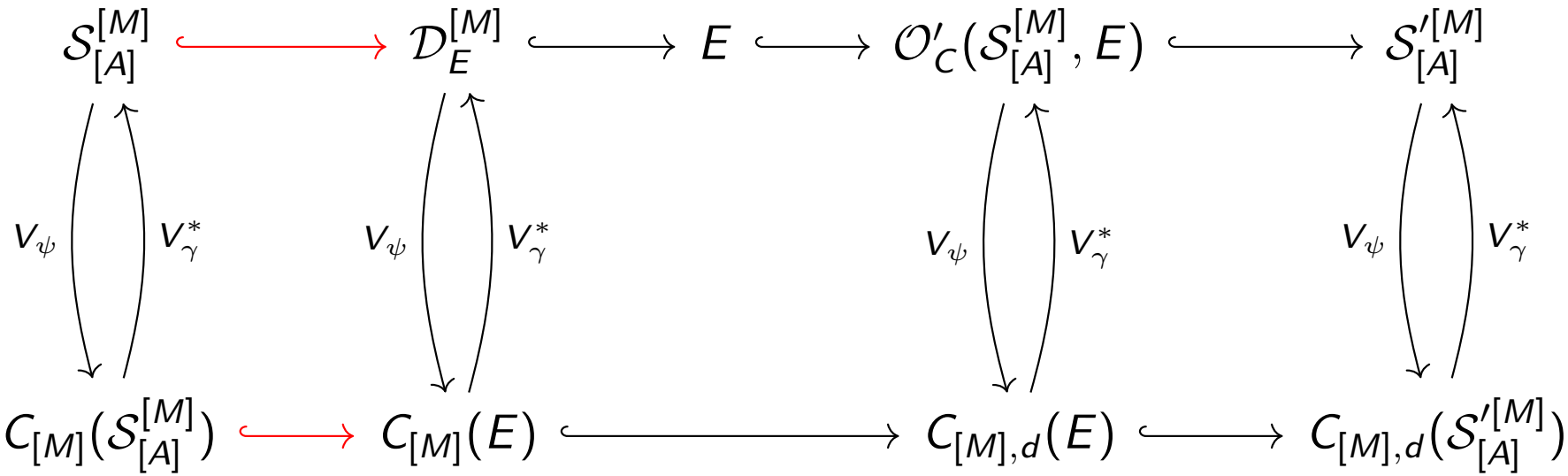
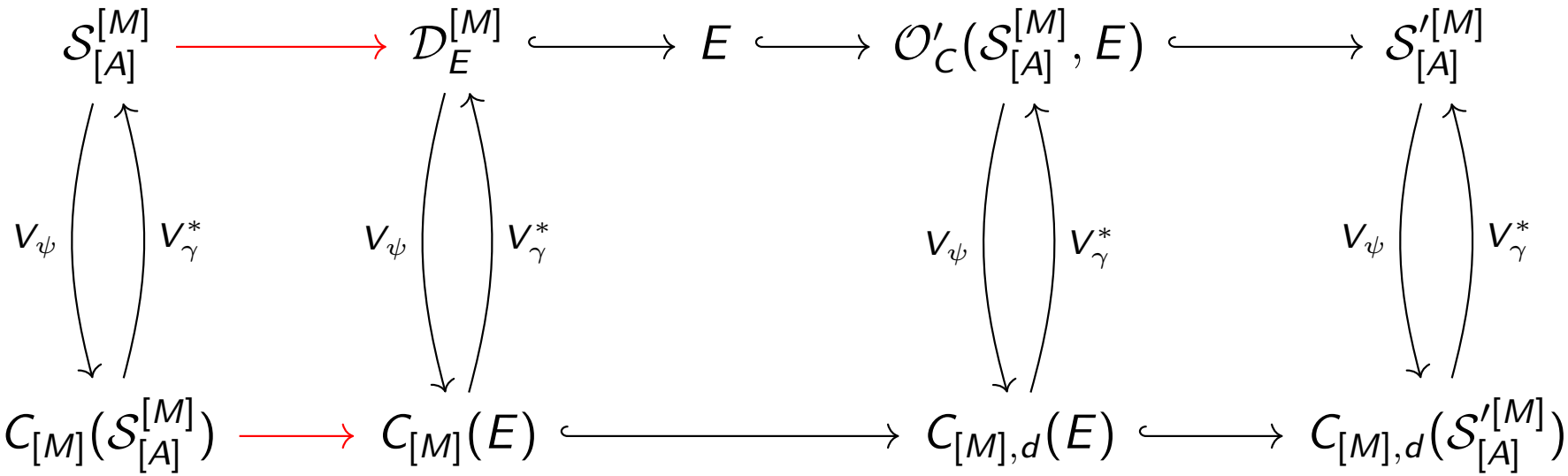
STFT – Characterizations

If E is a TMIB, we find the diagram



STFT – Characterizations

If E is a DTMIB, we find the diagram



Consequences – Topological properties

Let E be a TMIB or DTMIB.

Theorem

The space $\mathcal{D}_E^{(M)}$ is a quasinormable Fréchet space, and $\mathcal{D}_E^{\{M\}}$ is a complete (LB)-space.

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Let E be a TMIB.

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$(\mathcal{D}_E^{[M]})'_b = \mathcal{O}'_C(\mathcal{S}_{[A]}^{[M]}, E')$ as locally convex spaces.

Theorem

The mapping

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is a topological isomorphism.

Corollary

If E is reflexive, then so are $\mathcal{D}_E^{[M]}$ and $\mathcal{O}'_C(\mathcal{S}_{[A]}^{[M]}, E)$.

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Consequences – Structure of $\mathcal{O}'_C(\mathcal{S}^{[M]}_{[A]}, E)$

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Theorem

For any $f \in \mathcal{S}'^{[M]}_{[A]}$ we have

$$f * \varphi \in E, \qquad \forall \varphi \in \mathcal{S}^{[M]}_{[A]}$$

if and only if there exist $f_\alpha \in E$, $\alpha \in \mathbb{N}^d$, such that

$$f = \sum_{\alpha \in \mathbb{N}^d} f_\alpha^{(\alpha)}$$

and for some $\ell > 0$ (for every $\ell > 0$) we have that

$$\sup_{\alpha \in \mathbb{N}^d} \ell^{|\alpha|} M_\alpha \|f_\alpha\|_E < \infty$$

Application – Extension of convolution (1)

From now on we fix a TMIB or DTMIB E and assume M satisfies $(M.2)$.

For any $f = \sum_{\alpha \in \mathbb{N}^d} f_\alpha^{(\alpha)} \in \mathcal{O}'_C(\mathcal{S}^{[M]}_{[A]}, E)$ and $\psi \in \mathcal{D}^{[M]}_{L^1_{\omega_E}}$ we define the convolution

$$f * \psi := \sum_{\alpha \in \mathbb{N}^d} f_\alpha * \psi^{(\alpha)} \in \mathcal{D}^{[M]}_E$$

Theorem

The convolution mapping

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If E is a TMIB, then for any $f \in \mathcal{O}'_C(\mathcal{S}^{[M]}_{[A]}, E')$ and $g \in \mathcal{O}'_C(\mathcal{S}^{[M]}_{[A]}, E)$ we define the convolution

$$\langle f * g, \varphi \rangle := \langle f, \check{g} * \varphi \rangle, \qquad \varphi \in \mathcal{D}^{[M]}_{L^1_{\omega_E}},$$

so that $f * g \in (\mathcal{D}^{[M]}_{L^1_{\omega_E}})'_b = \mathcal{O}'_C(\mathcal{S}^{[M]}_{[A]}, L^\infty_{\omega_E})$.

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For any $f = \sum_{\alpha \in \mathbb{N}^d} f_\alpha^{(\alpha)} \in \mathcal{O}'_C(\mathcal{S}^{[M]}_{[A]}, E)$ and $g = \sum_{\beta \in \mathbb{N}^d} g_\beta^{(\beta)} \in \mathcal{O}'_C(\mathcal{S}^{[M]}_{[A]}, L^1_{\omega_E})$, we define the convolution

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