Weighted (PLB)-spaces of ultradifferentiable functions and multiplier spaces

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GF2022
Vienna, Austria, September 23, 2022

Introduction

The space $\mathcal{O}_M(\mathbb{R}^d)$

In his book *Théorie des distributions* L. Schwartz introduced the space

$$\mathcal{O}_M(\mathbb{R}^d) = \{\varphi \in C^\infty(\mathbb{R}^d) \mid \forall k\ \exists p : \sup_{|\alpha|\leq k, x\in\mathbb{R}^d} (1+|x|)^{-p}|\varphi^{(\alpha)}(x)| < \infty\}$$

which is a (PLB)-space, i.e. a countable projective limit of countable inductive limits of Banach spaces.

Equivalently, one may view it as the multiplier space

$$\mathcal{O}_M(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) \mid \varphi \cdot f \in \mathcal{S}(\mathbb{R}^d),\ \forall \varphi \in \mathcal{S}(\mathbb{R}^d)\}$$

and endow it with the topology induced by the embedding

$$\mathcal{O}_M(\mathbb{R}^d) \to \mathcal{L}(\mathcal{S}(\mathbb{R}^d), \mathcal{S}(\mathbb{R}^d)), \quad f \mapsto (\varphi \mapsto \varphi \cdot f)$$

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The topological invariants of $\mathcal{O}_M(\mathbb{R}^d)$

It is therefore non-trivial whether $\mathcal{O}_M(\mathbb{R}^d)$ is

* ultrabornological, i.e. if one is able to apply functional analytic tools such as De Wilde's open mapping and closed graph theorems;
* or even barrelled, i.e. if every absorbing, balanced, closed, convex subset of $\mathcal{O}_M(\mathbb{R}^d)$ is neighborhood of the origin.

- A. Grothendieck showed in his doctoral thesis (1955) that $\mathcal{O}_M(\mathbb{R}^d)$ is ultrabornological by:
  - $\hookrightarrow$ showing that $\mathcal{O}_M(\mathbb{R}^d)$ is isomorphic to a complemented subspace of $s\widehat{\otimes}_\pi s'$;
  - $\hookrightarrow$ directly verifying that $s\widehat{\otimes}_\pi s'$ is ultrabornological.
- More recently (2014), an alternative proof was given by J. Larcher and J. Wengenroth using homological methods.

The aim of this talk is to present analogues of these results in the setting of ultradifferentiable functions, both of Beurling and Roumieu type.

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References

This talk is based on collaborative work with Andreas Debrouwere (VUB).

We refer to the following works as the main influences of this talk:

S. Agethen, K. D. Bierstedt, J. Bonet, *Projective limits of weighted (LB)-spaces of continuous functions*, Arch. Math. **92** (2009), 384–398.

A. Debrouwere, L. Neyt, *Weighted (PLB)-spaces of ultradifferentiable functions and multiplier spaces*, Monatsh. Math **198** (2022), 31–60.

A. Debrouwere, L. Neyt, *A note on the barrelledness of weighted (PLB)-spaces of ultradifferentiable functions*, preprint, arXiv:2206.10252.

J. Wengenroth, *Derived functors in functional analysis*, Springer-Verlag, Berlin, 2003.

Topological invariants of (PLB)-spaces

(PLB)-spaces

Suppose $X_{n,N}$ are Banach spaces with norm $\|\cdot\|_{n,N}$, $n, N \in \mathbb{N}$, for which we have the continuous inclusions

$$X_{n+1,N} \subseteq X_{n,N} \subseteq X_{n,N+1}$$

We set

$$B_{n,N} = \{x \in X_{n,N} \mid \|x\|_{n,N} \leq 1\}$$

Then, for any $n \in \mathbb{N}$ we define the (LB)-space

$$X_n := \bigcup_{N\in\mathbb{N}} X_{n,N} = \varinjlim_{N\in\mathbb{N}} X_{n,N}$$

and the (PLB)-space

$$X := \bigcap_{n\in\mathbb{N}} X_n = \bigcap_{n\in\mathbb{N}} \bigcup_{N\in\mathbb{N}} X_{n,N} = \varprojlim_{n\in\mathbb{N}} \varinjlim_{N\in\mathbb{N}} X_{n,N}$$

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Sufficient condition for ultrabornologicity

Theorem (Braun, Vogt, 1997)

If

$$\forall n \in \mathbb{N}\ \exists m \geq n, N \in \mathbb{N}\ \forall k \geq m, M \geq N, \varepsilon > 0\ \exists K \geq M, C > 0$$
$$B_{m,M} \subseteq \varepsilon \cdot B_{n,N} + C \cdot B_{k,K}$$

then X is ultrabornological.

Necessary condition for barrelledness

We say X is *strongly reduced* if

$$\forall n \in \mathbb{N}\ \exists m \geq n : X_m \subseteq \overline{X}^{X_n}$$

Theorem (Vogt, 1987)

If X is strongly reduced and barrelled, then,

$$\forall n \in \mathbb{N}\ \exists m \geq n, N \in \mathbb{N}\ \forall k \geq m, M \geq N\ \exists K \geq M, C > 0$$
$$\forall y \in X_n' :\ \|y\|^*_{m,M} \leq C\left(\|y\|^*_{n,N} + \|y\|^*_{k,K}\right)$$

with $\|\cdot\|^*_{I,L}$ the dual norm of $\|\cdot\|_{I,L}$.

Summary

For a strongly reduced (PLB)-space X we thus have the following implications

$$\boxed{\begin{array}{c}\forall n \in \mathbb{N}\ \exists m \geq n, N \in \mathbb{N}\ \forall k \geq m, M \geq N, \varepsilon > 0\ \exists K \geq M, C > 0 \\ B_{m,M} \subseteq \varepsilon \cdot B_{n,N} + C \cdot B_{k,K}\end{array}}$$

$$\Downarrow$$

X is ultrabornological

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Weighted (PLB)-spaces of continuous functions

Definition

Let $v : \mathbb{R}^{2d} \to (0, \infty)$ be continuous. Then, we consider the Banach space

$$Cv(\mathbb{R}^{2d}) = \{f \in C(\mathbb{R}^{2d}) \mid \|f\|_v = \sup_{z \in \mathbb{R}^{2d}} v(z)|f(z)| < \infty\}.$$

Let $\mathcal{A} = (a_{n,N})_{n,N \in \mathbb{N}}$ be a double sequence of continuous functions $\mathbb{R}^{2d} \to (0, \infty)$ such that

$$1 \leq a_{n,N+1} \leq a_{n,N} \leq a_{n+1,N}$$

Then, we define the (PLB)-space

$$\mathcal{A}C(\mathbb{R}^{2d}) = \varprojlim_n \varinjlim_N Ca_{n,N}(\mathbb{R}^{2d})$$

$$= \left\{ f \in C(\mathbb{R}^{2d}) \mid \forall n \, \exists N : \sup_{z \in \mathbb{R}^{2d}} a_{n,N}(z)|f(z)| < \infty \right\}$$

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Topological invariants

Theorem (Agethen, Bierstedt, Bonet, 2009)

$$\forall n \in \mathbb{N}\ \exists m \geq n, N \in \mathbb{N}\ \forall k \geq m, M \geq N, \varepsilon > 0\ \exists K \geq M, C > 0\ \forall z \in \mathbb{R}^{2d}:$$
$$\frac{1}{a_{m,M}(z)} \leq \frac{\varepsilon}{a_{n,N}(z)} + \frac{C}{a_{k,K}(z)} \tag{Q}$$

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$\mathcal{A}C(\mathbb{R}^{2d})$ is ultrabornological $\Longrightarrow$ $\mathcal{A}C(\mathbb{R}^{2d})$ is barrelled

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(DN) and $(\overline{\overline{\Omega}})$

A family $\mathscr{V} = \{v_\lambda \mid \lambda \in (0,\infty)\}$ of continuous functions $v_\lambda : \mathbb{R}^d \to (0,\infty)$ is called a weight system if

$$1 \leq v_\lambda \leq v_\mu, \qquad \forall \mu \leq \lambda$$

$\mathscr{V}$ satisfies (DN) if

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Weight functions

A non-decreasing continuous function $\omega : \mathbb{R}^d \to (0, \infty)$ is called a weight function if $\omega(0) = 0$ and $\omega(t) \to \infty$ as $t \to \infty$.
We extend ω to $\mathbb{R}^d$ as the radial function $\omega(x) = \omega(|x|)$, $x \in \mathbb{R}^d$.

To any weight function ω we may associate the weight system

$$\mathscr{V}_\omega = \{e^{\frac{1}{\lambda}\omega} \mid \lambda \in (0, \infty)\}$$

and often also denote it by ω.

Then,

- $\mathscr{V}_\omega$ satisfies (DN).
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$$\mathscr{V}_\omega = \{e^{\frac{1}{\lambda}\omega} \mid \lambda \in (0, \infty)\}$$

and often also denote it by ω.

Then,

- $\mathscr{V}_\omega$ satisfies (DN).
- $\mathscr{V}_\omega$ does not satisfy $(\overline{\overline{\Omega}})$.

Specific case

To any two weight systems $\mathscr{V}$ and $\mathscr{W}$ we associate

$$\mathcal{A}_{(\mathscr{V},\mathscr{W})} = \left\{ \frac{v_{1/n}}{w_{1/N}} \mid n, N \in \mathbb{N} \right\}, \qquad \mathcal{A}_{\{\mathscr{V},\mathscr{W}\}} = \left\{ \frac{w_N}{v_n} \mid n, N \in \mathbb{N} \right\}$$

Lemma

Let ω be a weight function and $\mathscr{V}$ a weight system. Then, the following are equivalent:

(*i*) $\mathcal{A}_{(\mathscr{V}_\omega,\mathscr{V})}$ satisfies (Q); $\mathcal{A}_{\{\mathscr{V},\mathscr{V}_\omega\}}$ satisfies (Q).

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Weighted (PLB)-spaces of ultradifferentiable functions

BMT Weight functions

A weight function ω is called a BMT weight function if the following hold

(α) $\omega(2t) = O(\omega(t))$ as $t \to \infty$;

(γ) $\log t = o(\omega(t))$ as $t \to \infty$;

(δ) $\phi : [0, \infty) \to [0, \infty)$, $\phi(x) = \omega(e^x)$, is convex.

The Young conjugate of ϕ is defined as

$$\phi^* : [0, \infty) \to [0, \infty), \quad \phi^*(y) = \sup_{x \geq 0}\{xy - \phi(x)\}$$

Example – Gevrey sequences

For $s > 0$ we may consider $\omega(x) = x^{\frac{1}{s}}$. Then, for some $k > 0$,

$$e^{\phi^*(n)} \sim k^n (n!)^s, \qquad \text{as } n \to \infty$$

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The Gelfand-Shilov spaces (1)

We now fix a BMT weight function ω and a weight system $\mathscr{V}$.

For any $\lambda > 0$, we consider the Banach space

$$\mathcal{S}_{v_\lambda}^{\omega,\lambda} = \left\{ \varphi \in C^\infty(\mathbb{R}^d) : \sup_{(\alpha,x)\in\mathbb{N}^d\times\mathbb{R}^d} |\varphi^{(\alpha)}(x)| v_\lambda(x) e^{-\frac{1}{\lambda}\phi^*(\lambda|\alpha|)} < \infty \right\}.$$

We define the Gelfand-Shilov spaces *of Beurling and Roumieu type* as

$$\mathcal{S}_{(\mathscr{V})}^{(\omega)} = \varprojlim_{\lambda\to 0^+} \mathcal{S}_{v_\lambda}^{\omega,\lambda}, \qquad \mathcal{S}_{\{\mathscr{V}\}}^{\{\omega\}} = \varinjlim_{\lambda\to\infty} \mathcal{S}_{v_\lambda}^{\omega,\lambda}$$

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The Gelfand-Shilov spaces (2)

Suppose η is another BMT weight function. Then,

$$\varphi \in \mathcal{S}_{(\eta)}^{(\omega)} \iff \forall \lambda : \quad \sup_{(x,\xi)\in\mathbb{R}^{2d}} |\varphi(x)| e^{\frac{1}{\lambda}\eta(x)} + |\widehat{\varphi}(\xi)| e^{\frac{1}{\lambda}\omega(\xi)} < \infty$$

and

$$\varphi \in \mathcal{S}_{\{\eta\}}^{\{\omega\}} \iff \exists \lambda : \quad \sup_{(x,\xi)\in\mathbb{R}^{2d}} |\varphi(x)| e^{\frac{1}{\lambda}\eta(x)} + |\widehat{\varphi}(\xi)| e^{\frac{1}{\lambda}\omega(\xi)} < \infty$$

where $\widehat{\varphi}$ denotes the Fourier transform of φ.

For $r, s > 0$, we find

- the classical Gelfand-Shilov space of Beurling type $\Sigma_s^r = \mathcal{S}_{(|\cdot|^{1/s})}^{(|\cdot|^{1/r})}$ which is non-trivial if and only if $r + s > 1$.
- the classical Gelfand-Shilov space of Roumieu type $\mathcal{S}_s^r = \mathcal{S}_{\{|\cdot|^{1/s}\}}^{\{|\cdot|^{1/r}\}}$ which is non-trivial if and only if $r + s \geq 1$.

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The ultradifferentiable (PLB)-spaces

For any $h, \lambda > 0$, we consider the Banach space

$$\mathcal{Z}_{v_\lambda}^{\omega,h} = \left\{ \varphi \in C^\infty(\mathbb{R}^d) : \sup_{(\alpha,x)\in\mathbb{N}^d\times\mathbb{R}^d} \frac{|\varphi^{(\alpha)}(x)|}{v_\lambda(x)} e^{-\frac{1}{h}\phi^*(h|\alpha|)} < \infty \right\}.$$

We define the weighted ultradifferentiable (PLB)-spaces *of Beurling and Roumieu type* as

$$\mathcal{Z}_{(\mathcal{V})}^{(\omega)} = \varprojlim_{h\to 0^+} \varinjlim_{\lambda\to 0^+} \mathcal{Z}_{v_\lambda}^{\omega,h}, \qquad \mathcal{Z}_{\{\mathcal{V}\}}^{\{\omega\}} = \varprojlim_{\lambda\to\infty} \varinjlim_{h\to\infty} \mathcal{Z}_{v_\lambda}^{\omega,h}$$

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Main result

Theorem (Debrouwere, N., 2022)

Suppose $\mathcal{S}^{[\omega]}_{[\mathscr{V}]} \neq \{0\}$ and $\mathscr{V}$ satisfies some conditions ([M] and [N]). Then, the following are equivalent:

(*i*) $\mathscr{V}$ satisfies (DN) (Beurling case); $\mathscr{V}$ satisfies $(\overline{\overline{\Omega}})$ (Roumieu case).

(*ii*) $\mathcal{Z}^{[\omega]}_{[\mathscr{V}]}$ is ultrabornological.

(*iii*) $\mathcal{Z}^{[\omega]}_{[\mathscr{V}]}$ is barrelled.

Proof of $(i) \Rightarrow (ii)$

Our main tool is the use of the short-time Fourier transform (STFT) with a non-trivial window $\psi \in \mathcal{S}^{[\omega]}_{[\mathscr{V}]}$:

$$f \mapsto V_\psi f(x,\xi) := \int_{\mathbb{R}^d} f(t)\overline{\psi(t-x)}e^{-2\pi i \xi \cdot t}dt$$

We have

$$\varphi \in \mathcal{Z}^{(\omega)}_{(\mathscr{V})} \iff V_\psi \varphi \in \mathcal{A}_{(\mathscr{V}_\omega,\mathscr{V})}C(\mathbb{R}^{2d})$$

and

$$\varphi \in \mathcal{Z}^{\{\omega\}}_{\{\mathscr{V}\}} \iff V_\psi \varphi \in \mathcal{A}_{\{\mathscr{V},\mathscr{V}_\omega\}}C(\mathbb{R}^{2d})$$

and the mappings of the (adjoint) STFT are continuous

Consequently

$$\mathcal{Z}^{(\omega)}_{(\mathscr{V})} \cong \mathcal{A}_{(\mathscr{V}_\omega,\mathscr{V})}C(\mathbb{R}^{2d})/\ker V_\psi^* \quad \text{and} \quad \mathcal{Z}^{\{\omega\}}_{\{\mathscr{V}\}} \cong \mathcal{A}_{\{\mathscr{V},\mathscr{V}_\omega\}}C(\mathbb{R}^{2d})/\ker V_\psi^*$$

By the lemma, $\mathcal{A}_{(\mathscr{V}_\omega,\mathscr{V})}C(\mathbb{R}^{2d})$ and $\mathcal{A}_{\{\mathscr{V},\mathscr{V}_\omega\}}C(\mathbb{R}^{2d})$ are ultrabornological.

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Proof of $(iii) \Rightarrow (i)$

Suppose $\psi \in \mathcal{S}_{\lambda_0}^{\omega,h_0}$. Using the result of Vogt, we show that if $\mathcal{Z}_{\{\mathscr{V}\}}^{\{\omega\}}$ is barrelled, then,

$$\forall \lambda > \lambda_0 \ \exists \mu > \lambda, h > h_0 \ \forall \kappa > \mu, k > h \ \exists \ell > k, C > 0 \ \forall f \in C_{v_\lambda} :$$
$$\|V_\psi f\|_{v_\mu \otimes e^{-\frac{1}{k}\omega}} \leq C \left(\|V_\psi f\|_{v_\lambda \otimes e^{-\frac{1}{h}\omega}} + \|V_\psi f\|_{v_\kappa \otimes e^{-\frac{1}{\ell}\omega}} \right)$$

and a similar result holds for the case of $\mathcal{Z}_{(\mathscr{V})}^{(\omega)}$.

By choosing $f = f_{x,\xi} = \psi(\cdot - x)e^{2\pi i \xi \cdot}$ for $x, \xi \in \mathbb{R}^d$ and using the lemma we find that

$$\mathcal{Z}_{(\mathscr{V})}^{(\omega)} \text{ is barrelled } \Longrightarrow \mathcal{A}_{(\mathscr{V}_\omega, \mathscr{V})} \text{ satisfies } (wQ) \Longrightarrow \mathscr{V} \text{ satisfies (DN)}$$

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Multiplier spaces (1)

We define the multiplier space of $\mathcal{S}^{[\omega]}_{[\mathscr{V}]}$ as

$$\mathcal{O}_M(\mathcal{S}^{[\omega]}_{[\mathscr{V}]}) = \{f \in \mathcal{S}'^{[\omega]}_{[\mathscr{V}]} : \varphi \cdot f \in \mathcal{S}^{[\omega]}_{[\mathscr{V}]},\ \forall \varphi \in \mathcal{S}^{[\omega]}_{[\mathscr{V}]}\}$$

with the topology induced by the embedding

$$\mathcal{O}_M(\mathcal{S}^{[\omega]}_{[\mathscr{V}]}) \to \mathcal{L}_b(\mathcal{S}^{[\omega]}_{[\mathscr{V}]}, \mathcal{S}^{[\omega]}_{[\mathscr{V}]}) : \quad f \mapsto (\varphi \mapsto \varphi \cdot f)$$

Theorem (Debrouwere, N., 2022)

Suppose that $\mathcal{S}^{[\omega]}_{[\mathscr{V}]} \neq 0$ and $\mathscr{V}$ satisfies some conditions ([M], [N], and [Sq]). Then,

$$\mathcal{O}_M(\mathcal{S}^{[\omega]}_{[\mathscr{V}]}) = \mathcal{Z}^{[\omega]}_{[\mathscr{V}]}$$

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Multiplier spaces (2)

Theorem (Debrouwere, N., 2022)

Suppose ω and η are BMT weight functions such that $\mathcal{S}^{[\omega]}_{[\eta]} \neq \{0\}$.

- $\mathcal{O}_M(\mathcal{S}^{(\omega)}_{(\eta)})$ is ultrabornological.
- $\mathcal{O}_M(\mathcal{S}^{\{\omega\}}_{\{\eta\}})$ is not barrelled.

Corollary (Debrouwere, N., 2022)

Let $r, s > 0$.

- If $r + s > 1$, then, $\mathcal{O}_M(\Sigma^r_s)$ is ultrabornological.
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