The structure of ultradistributional quasiasymptotics

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By M_p we will denote a sequence of positive numbers with $M_0 = 1$ and associate to it the sequence $M_p^* = M_p/p!$. We will use the conditions $(M.1)$, $(M.1)^*$, $(M.2)$, $(M.3)'$, $(M.3)$ as in Komatsu. The **associated function** of M_p is

$$M(t) = \sup_{p \in \mathbb{N}} \log \frac{t^p}{M_p}$$

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For any compact $K \Subset \mathbb{R}^d$ and $h > 0$ we let $\mathcal{D}_{K,h}^{\{M_p\}}$ be Banach space of smooth functions ϕ with support in K that satisfy the norm $\|\phi\|_{K,h} = \sup_{x \in K, \alpha \in \mathbb{N}^d} |\phi^{(\alpha)}(x)|/h^{|\alpha|} M_{|\alpha|}$. Then we let

$$D^{(M_p)}(\mathbb{R}^d) = \varinjlim_{K \Subset \mathbb{R}^d} \varprojlim_{h \to 0+} \mathcal{D}_{K,h}^{\{M_p\}}$$

$$D^{\{M_p\}}(\mathbb{R}^d) = \varinjlim_{K \Subset \mathbb{R}^d} \varinjlim_{h \to \infty} \mathcal{D}_{K,h}^{\{M_p\}}$$

The common notation for (M_p) and $\{M_p\}$ will be $*$

The simplify notation between the Beurling and Roumieu case we will use the set

$$\mathfrak{R}^{(M_p)} := \{(\ell_p)_{p\in\mathbb{N}} : \ell_p = \ell \text{ for some } \ell > 0\},$$
$$\mathfrak{R}^{\{M_p\}} := \{(\ell_p)_{p\in\mathbb{N}} : \ell_p \nearrow \infty \text{ and } \ell_p > 0, \forall p \in \mathbb{N}\},$$

For any $(\ell_p) \in \mathfrak{R}^*$ we set $L_p = \prod_{k=1}^{p} \ell_k$ and denote the associated function of $L_p M_p$ as $M_{\ell_p}(t)$.

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For two weight sequence A_p, B_p and any $(\ell_p) \in \mathfrak{R}^*$, $\mathcal{S}^{A_p(\ell_p)}_{B_p,(\ell_p)}$ denotes the (B)-space of all smooth ϕ s.t. $\sup_{\alpha,\beta\in\mathbb{N}^d} \left\|x^\beta \phi^{(\alpha)}\right\|_\infty / L_{|\alpha|}A_{|\alpha|}L_{|\beta|}B_{|\beta|} < \infty$. Then

$$\mathcal{S}^\star_\dagger(\mathbb{R}^d) = \varprojlim_{(\ell_p)\in\mathfrak{R}^*} \mathcal{S}^{A_p(\ell_p)}_{B_p,(\ell_p)}$$

If $M_p = A_p = B_p$ we simply denote $\mathcal{S}^*_*(\mathbb{R}^d)$ as $\mathcal{S}^*(\mathbb{R}^d)$ and elements in $\mathcal{S}^{*\prime}(\mathbb{R}^d)$ are called **tempered ultradistributions**.

Definition

Let $\mathcal{F}$ be a space of smooth functions, closed under dilation, and let $f \in \mathcal{F}'$. Then f has **quasiasymptotic behavior** $g \in \mathcal{F}'$ with respect to the gauge function ρ if

$$f(\lambda x) \sim \rho(\lambda) g(x)$$

holds in $\mathcal{F}'$, as $\lambda \to \infty$ (resp. $\lambda \to 0+$).

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Lemma for $\mathcal{F} = \mathcal{D}(\mathbb{R})$ or $\mathcal{D}^*(\mathbb{R})$

1. $\rho(\lambda) = \lambda^\alpha L(\lambda)$ for some **slowly varying function** L
2. g is **homogeneous** of degree α. If $\alpha \notin \mathbb{Z}_-$:

$$g(x) = c_+ x_+^\alpha + c_- x_-^\alpha$$

If $\alpha = -k$, $k \in \mathbb{Z}_+$:

$$g(x) = c_1 x^{-k} + c_2 \delta^{(k-1)}(x)$$

Known results for $\mathcal{D}'(\mathbb{R})$

Suppose that

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$$\lim_{x \to \infty} \frac{F(\pm x)}{|x|^{\alpha+m} L(|x|)} \text{ exists}$$

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If $\alpha = -k$, there exists a m-primitive F of f, $m \geq k$, such that $\forall a > 0$:

$$\lim_{x \to \infty} \frac{a^{k-m} F(ax) - (-1)^{m-k} F(-x)}{x^{m-k} L(x)} \text{ exists}$$

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2. f is an element of $\mathcal{S}'(\mathbb{R})$ and the same asymptotic holds there

Proposition [$(M.1)$, $(M.3)'$]

For any $f \in \mathcal{E}^{*\prime}(\mathbb{R})$ there exist $\mu_n \in \mathbb{C}$, $n \in \mathbb{N}$, such that

$$f(\lambda x) \sim \sum_{n=0}^{\infty} \frac{(-1)^n \mu_n \delta^{(n)}(x)}{n! \lambda^{n+1}}, \text{ in } \mathcal{E}^{*\prime}(\mathbb{R})$$

Hence for $\alpha > -1$ quasiasymptotic behavior is **local**!

Strategy for structure theorems:

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2. If $\alpha > -1$: use locality and substitution of the exponential to turn quasiasymptotic into **S-asymptotic**
3. If $\alpha = -1$: multiply by x and use the previously obtained structure theorem

New results for $\mathcal{D}^{*\prime}(\mathbb{R})$ - Case: $\alpha \notin \mathbb{Z}_-$

Theorem [$(M.1)$, $(M.2)$, $(M.3)$]

Let k be the smallest non-negative integer such that $-(k+1) < \alpha$. Then

$$f(\lambda x) \sim \lambda^{\alpha} L(\lambda) g(x) \qquad \text{in } \mathcal{D}^{*\prime}(\mathbb{R})$$

if and only if there exist continuous f_m on $\mathbb{R}$, $m \geq k$, such that

$$f = \sum_{m=k}^{\infty} f_m^{(m)}$$

the limits

$$\lim_{x \to \pm\infty} \frac{f_m(x)}{|x|^{\alpha+m} L(|x|)} \text{ exist}$$

and for some $\ell > 0$ (any $\ell > 0$) there is a $C = C_\ell > 0$ such that

$$|f_m(x)| \leq C \frac{\ell^m}{M_m} (1 + |x|)^{\alpha+m} L(|x|)$$

New results for $\mathcal{D}^{*\prime}(\mathbb{R})$ - Case: $\alpha = -k$

Theorem [$(M.1)$, $(M.2)$, $(M.3)$]

$$f(\lambda x) \sim \lambda^{-k} L(\lambda) g(x) \qquad \text{in } \mathcal{D}^{*\prime}(\mathbb{R})$$

if and only if there exist continuous f_m on $\mathbb{R}$, $m \geq k-1$, such that

$$f = f_{k-1} + \sum_{m=k}^{\infty} f_m^{(m)}$$

that satisfy the **previous conditions** and in addition

$$\lim_{x \to \infty} \frac{1}{L(x)} \int_{-x}^{x} f_{k-1}(t)dt$$

exists.

Extension to $\mathcal{S}^{\star\prime}_{\dagger}(\mathbb{R})$

Theorem [$(M.1)$, $(M.2)$, $(M.3)$]

Suppose $A_p B_p \leq C\mu^p M_p$. If $f \in \mathcal{D}^{*\prime}(\mathbb{R})$ and

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then $f \in S^{\star\prime}_{\dagger}(\mathbb{R})$ and the quasiasymptotic behavior holds in $S^{\star\prime}_{\dagger}(\mathbb{R})$.

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The structure of quasiasymptotic behavior in $\mathcal{S}^{*\prime}(\mathbb{R})$ is an **open problem**.

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↳ Alternative: find equivalent definitions using Tauberian theorems of certain integrals transforms

Laplace transform on a cone

$\Gamma \subseteq \mathbb{R}^d$ is called a **cone** if $\lambda\Gamma \subseteq \Gamma$ for every $\lambda > 0$, i.e. Γ is closed under dilation

A cone Γ is called **acute** if $C := \{w \in \mathbb{R}^d : u \cdot w > 0, \forall u \in \Gamma\}$ is nonempty

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Definition

For any $f \in \mathcal{S}'^*_\Gamma$, i.e. $\operatorname{supp} f \subseteq \Gamma$, if there is a η such that $\eta(\xi) = 1$ on a neighbourhood of Γ and $\eta(\cdot)e^{iz\cdot} \in \mathcal{S}^*(\mathbb{R}^d)$ for every $z \in T^C = \mathbb{R}^d + iC$, then the **Laplace transform** of f is defined as

$$\mathcal{L}\{f; z\} := \left\langle f(x), \eta(x)e^{iz\cdot x} \right\rangle, \qquad \forall z \in T^C$$

This definition is independent of η and exists when non-quasianalyticity is assumed.

Since $\mathcal{L}\{f;z\}$ is analytic on T^C, the Laplace transform is injective so that for a family $\{f_\lambda\}_{\lambda>0} \subseteq \mathcal{S}'^*_\Gamma$:

$$\lim_{\lambda\to\infty} f_\lambda = g \Leftrightarrow \begin{cases} \lim\limits_{\lambda\to\infty} \mathcal{L}\{f_\lambda;z\} = \mathcal{L}\{g;z\} \\ \{f_\lambda\}_{\lambda>0} \text{ is bounded} \end{cases}$$

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Using the **short-time Fourier transform** we showed

Proposition [$(M.1)^*$, $(M.2)$, $(M.3)'$]

Let $B \subseteq \mathcal{S}'^*(\mathbb{R}^d)$. Then B is **bounded** if an only if there exists $(\ell_p) \in \mathfrak{R}^*$ such that for every $\varphi \in \mathcal{D}^*(\mathbb{R}^d)$

$$e^{-M_{\ell_p}(|x|)}\,|(f*\varphi)(x)| < \infty$$

Theorem [$(M.1)^*$, $(M.2)$, $(M.3)'$]

Let $\rho : \mathbb{R}_+ \to \mathbb{R}_+$. Then for $\{f_\lambda\}_{\lambda>0} \subseteq \mathcal{S}_\Gamma^{*\prime}$ and $g \in \mathcal{S}_\Gamma^{*\prime}$

$$f_\lambda(x) \sim \rho(\lambda) g(x), \qquad \lambda \to \infty$$

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if and only if

1. for some open $\emptyset \neq U \subseteq C$

$$\lim_{\lambda\to\infty} \frac{1}{\rho(\lambda)} \mathcal{L}\{f_\lambda; iy\} = \mathcal{L}\{g; iy\}, \qquad \forall y \in U$$

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2. there is $(\ell_p) \in \mathfrak{R}^*$ such that for every $\varepsilon > 0$ there is a $L = L_\varepsilon > 0$ so that $\forall z = x + iy \in T^C$, $\lambda > 0$:

$$|\mathcal{L}\{f_\lambda; z\}| \leq L\rho(\lambda) \exp\left(\varepsilon|y| + M_{\ell_p}(|z|) + M^*_{\ell_p}\left(\frac{1}{\Delta_C(y)}\right)\right)$$

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If the M_p satisfy $(M.1)$, $(M.2)$, $(M.3)$ and if $\operatorname{int}\Gamma \neq \emptyset$ then we may put $\varepsilon = 0$

Theorem $[(M.1)^*, (M.2), (M.3)']$

If $f, g \in \mathcal{S}'^*_{[0,\infty)}$ then the following are equivalent:

A. $f(\lambda x) \sim \lambda^\alpha L(\lambda) g(x)$ in $\mathcal{S}'^*(\mathbb{R})$

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B.
   1. $\lim_{y \to 0+} \frac{y^{\alpha+1}}{L(1/y)} \mathcal{L}\{f; iy\} = C$
   2. there exist $(\ell_p) \in \mathfrak{R}^*$, $L > 0$ and $r_0 > 0$ such that

$$\left| \frac{r^{\alpha+1}}{L(1/r)} \mathcal{L}\left\{f; re^{i\theta}\right\} \right| \leq L \exp\left(M^*_{\ell_p}\left(\frac{1}{\sin\theta} \right) \right), \quad \forall 0 < \theta < \pi, 0 < r < r_0$$

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C. for every $\phi \in \mathcal{D}^*(\mathbb{R})$ (for some $\phi \in \mathcal{D}^*(\mathbb{R})$ such that $\int \phi \neq 0$)

$$(f * \phi)(\lambda x) \sim \lambda^\alpha L(\lambda) \left(\int \phi \right) g(x) \qquad \text{in } \mathcal{S}'^*(\mathbb{R})$$

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D. for any δ-sequence $\{\delta_m\}_{m \in \mathbb{N}}$

$$(f * \delta_m)(\lambda x) \sim \lambda^\alpha L(\lambda) g(x) \qquad \text{in } \mathcal{S}'^*(\mathbb{R})$$

uniformly in $m \in \mathbb{N}$